

User's Guide
to
the PARI library

(version 2.11.4)

The PARI Group

Institut de Mathématiques de Bordeaux, UMR 5251 du CNRS.
Université de Bordeaux, 351 Cours de la Libération
F-33405 TALENCE Cedex, FRANCE
e-mail: `pari@math.u-bordeaux.fr`

Home Page:
<http://pari.math.u-bordeaux.fr/>

Copyright © 2000–2018 The PARI Group

Permission is granted to make and distribute verbatim copies of this manual provided the copyright notice and this permission notice are preserved on all copies.

Permission is granted to copy and distribute modified versions, or translations, of this manual under the conditions for verbatim copying, provided also that the entire resulting derived work is distributed under the terms of a permission notice identical to this one.

PARI/GP is Copyright © 2000–2018 The PARI Group

PARI/GP is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation. It is distributed in the hope that it will be useful, but **WITHOUT ANY WARRANTY WHATSOEVER**.

Table of Contents

Chapter 4: Programming PARI in Library Mode	13
4.1 Introduction: initializations, universal objects	13
4.2 Important technical notes	14
4.2.1 Backward compatibility	14
4.2.2 Types	14
4.2.3 Type recursivity	15
4.2.4 Variations on basic functions	15
4.2.5 Portability: 32-bit / 64-bit architectures	16
4.2.6 Using <code>malloc</code> / <code>free</code>	17
4.3 Garbage collection	17
4.3.1 Why and how	17
4.3.2 Variants	20
4.3.3 Examples	20
4.3.4 Comments	24
4.4 Creation of PARI objects, assignments, conversions	24
4.4.1 Creation of PARI objects	24
4.4.2 Sizes	26
4.4.3 Assignments	26
4.4.4 Copy	27
4.4.5 Clones	27
4.4.6 Conversions	28
4.5 Implementation of the PARI types	28
4.5.1 Type <code>t_INT</code> (integer)	29
4.5.2 Type <code>t_REAL</code> (real number)	30
4.5.3 Type <code>t_INTMOD</code>	31
4.5.4 Type <code>t_FRAC</code> (rational number)	31
4.5.5 Type <code>t_FFELT</code> (finite field element)	31
4.5.6 Type <code>t_COMPLEX</code> (complex number)	31
4.5.7 Type <code>t_PADIC</code> (p -adic numbers)	31
4.5.8 Type <code>t_QUAD</code> (quadratic number)	32
4.5.9 Type <code>t_POLMOD</code> (polmod)	32
4.5.10 Type <code>t_POL</code> (polynomial)	32
4.5.11 Type <code>t_SER</code> (power series)	33
4.5.12 Type <code>t_RFRAC</code> (rational function)	33
4.5.13 Type <code>t_QFR</code> (indefinite binary quadratic form)	33
4.5.14 Type <code>t_QFI</code> (definite binary quadratic form)	34
4.5.15 Type <code>t_VEC</code> and <code>t_COL</code> (vector)	34
4.5.16 Type <code>t_MAT</code> (matrix)	34
4.5.17 Type <code>t_VECSMALL</code> (vector of small integers)	34
4.5.18 Type <code>t_STR</code> (character string)	34
4.5.19 Type <code>t_ERROR</code> (error context)	34
4.5.20 Type <code>t_CLOSURE</code> (closure)	34
4.5.21 Type <code>t_INFINITY</code> (infinity)	34
4.5.22 Type <code>t_LIST</code> (list)	34
4.6 PARI variables	35
4.6.1 Multivariate objects	35

4.6.2	Creating variables	35
4.6.3	Comparing variables	37
4.7	Input and output	37
4.7.1	Input	37
4.7.2	Output to screen or file, output to string	38
4.7.3	Errors	39
4.7.4	Warnings	40
4.7.5	Debugging output	40
4.7.6	Timers and timing output	41
4.8	Iterators, Numerical integration, Sums, Products	42
4.8.1	Iterators	42
4.8.2	Iterating over primes	43
4.8.3	Numerical analysis	44
4.9	Catching exceptions	45
4.9.1	Basic use	45
4.9.2	Advanced use	45
4.10	A complete program	46
Chapter 5:	Technical Reference Guide: the basics	49
5.1	Initializing the library	49
5.1.1	General purpose	49
5.1.2	Technical functions	50
5.1.3	Notions specific to the GP interpreter	52
5.1.4	Public callbacks	53
5.1.5	Configuration variables	54
5.1.6	Utility functions	54
5.1.7	Saving and restoring the GP context	55
5.1.8	GP history	55
5.2	Handling GENs	55
5.2.1	Allocation	56
5.2.2	Length conversions	57
5.2.3	Read type-dependent information	58
5.2.4	Eval type-dependent information	59
5.2.5	Set type-dependent information	60
5.2.6	Type groups	61
5.2.7	Accessors and components	61
5.3	Global numerical constants	62
5.3.1	Constants related to word size	62
5.3.2	Masks used to implement the GEN type	62
5.3.3	$\log 2$, π	63
5.4	Iterating over small primes, low-level interface	63
5.5	Handling the PARI stack	64
5.5.1	Allocating memory on the stack	64
5.5.2	Stack-independent binary objects	65
5.5.3	Garbage collection	66
5.5.4	Garbage collection: advanced use	67
5.5.5	Debugging the PARI stack	68
5.5.6	Copies	68
5.5.7	Simplify	68
5.6	The PARI heap	69

5.6.1	Introduction	69
5.6.2	Public interface	69
5.6.3	Implementation note	69
5.7	Handling user and temp variables	70
5.7.1	Low-level	70
5.7.2	User variables	70
5.7.3	Temporary variables	70
5.8	Adding functions to PARI	71
5.8.1	Nota Bene	71
5.8.2	Coding guidelines	71
5.8.3	GP prototypes, parser codes	72
5.8.4	Integration with <code>gp</code> as a shared module	74
5.8.5	Library interface for <code>install</code>	75
5.8.6	Integration by patching <code>gp</code>	75
5.9	Globals related to PARI configuration	76
5.9.1	PARI version numbers	76
5.9.2	Miscellaneous	76
Chapter 6:	Arithmetic kernel: Level 0 and 1	77
6.1	Level 0 kernel (operations on ulongs)	77
6.1.1	Micro-kernel	77
6.1.2	Modular kernel	78
6.1.3	Modular kernel with “precomputed inverse”	79
6.1.4	Switching between <code>FL_xxx</code> and standard operators	80
6.2	Level 1 kernel (operations on longs, integers and reals)	81
6.2.1	Creation	81
6.2.2	Assignment	82
6.2.3	Copy	83
6.2.4	Conversions	83
6.2.5	Integer parts	84
6.2.6	2-adic valuations and shifts	85
6.2.7	From <code>t_INT</code> to bits or digits in base 2^k and back	86
6.2.8	Integer valuation	86
6.2.9	Generic unary operators	87
6.2.10	Comparison operators	88
6.2.11	Generic binary operators	89
6.2.12	Exact division and divisibility	92
6.2.13	Division with integral operands and <code>t_REAL</code> result	92
6.2.14	Division with remainder	93
6.2.15	Modulo to longs	94
6.2.16	Powering, Square root	95
6.2.17	GCD, extended GCD and LCM	96
6.2.18	Continued fractions and convergents	96
6.2.19	Pseudo-random integers	96
6.2.20	Modular operations	97
6.2.21	Extending functions to vector inputs	99
6.2.22	Miscellaneous arithmetic functions	100
Chapter 7:	Level 2 kernel	103
7.1	Naming scheme	103
7.2	Coefficient ring	105

7.3 Modular arithmetic	106
7.3.1 FpC / FpV, FpM	106
7.3.2 Flc / Flv, Flm	110
7.3.3 F2c / F2v, F2m	112
7.3.4 FlxqV, FlxqC, FlxqM	114
7.3.5 FpX	114
7.3.6 FpXQ, Fq	118
7.3.7 FpXQ	120
7.3.8 Fq	120
7.3.9 FpXn	122
7.3.10 FpXC, FpXM	122
7.3.11 FpXX, FpXY	123
7.3.12 FpXQX, FqX	123
7.3.13 FpXQXn, FqXn	126
7.3.14 FpXQXQ, FqXQ	126
7.3.15 Flx	129
7.3.16 FlxV	133
7.3.17 FlxM	134
7.3.18 FlxT	134
7.3.19 Flxq	134
7.3.20 FlxX	135
7.3.21 FlxqX	136
7.3.22 FlxqXQ	138
7.3.23 F2x	139
7.3.24 F2xq	141
7.3.25 F2xqV, F2xqM	142
7.3.26 F2xX	142
7.3.27 F2xXV/F2xXC	143
7.3.28 F2xqX	143
7.3.29 F2xqXQ	144
7.3.30 Functions returning objects with <code>t_INTMOD</code> coefficients	144
7.3.31 Slow Chinese remainder theorem over \mathbf{Z}	145
7.3.32 Fast remainders	147
7.3.33 Fast Chinese remainder theorem over \mathbf{Z}	148
7.3.34 Rational reconstruction	149
7.3.35 \mathbf{Z}_p	149
7.3.36 \mathbf{Z}_pX	150
7.3.37 \mathbf{Z}_pXQ	151
7.3.38 \mathbf{Z}_q	151
7.3.39 \mathbf{Z}_pXQM	151
7.3.40 \mathbf{Z}_pXQX	152
7.3.41 \mathbf{Z}_qX	152
7.3.42 Other p -adic functions	152
7.3.43 Conversions involving single precision objects	154
7.4 Higher arithmetic over \mathbf{Z} : primes, factorization	157
7.4.1 Pure powers	157
7.4.2 Factorization	158
7.4.3 Coprime factorization	160
7.4.4 Checks attached to arithmetic functions	160

7.4.5 Incremental integer factorization	161
7.4.6 Integer core, squarefree factorization	162
7.4.7 Primes, primality and compositeness tests	162
7.4.8 Iterators over primes	164
7.5 Integral, rational and generic linear algebra	164
7.5.1 ZC / ZV, ZM	164
7.5.2 QM	168
7.5.3 Qevproj	168
7.5.4 zv, zm	169
7.5.5 ZMV / zmV (vectors of ZM/zm)	169
7.5.6 QC / QV, QM	170
7.5.7 RgC / RgV, RgM	170
7.5.8 ZG	174
7.5.9 Blackbox linear algebra	175
7.5.10 Obsolete functions	176
7.6 Integral, rational and generic polynomial arithmetic	176
7.6.1 ZX	176
7.6.2 Resultants	179
7.6.3 ZXV	179
7.6.4 ZXT	179
7.6.5 ZXQ	180
7.6.6 ZXn	180
7.6.7 ZXQM	180
7.6.8 ZXQX	180
7.6.9 ZXX	180
7.6.10 QX	181
7.6.11 QXQ	181
7.6.12 QXQM	182
7.6.13 zx	183
7.6.14 RgX	183
7.6.15 RgXn	188
7.6.16 RgXnV	188
7.6.17 RgXQ	189
7.6.18 RgXQV, RgXQC	189
7.6.19 RgXQM	190
7.6.20 RgXQX	190
Chapter 8: Black box algebraic structures	191
8.1 Black box groups	191
8.1.1 Black box groups with pairing	193
8.1.2 Functions returning black box groups	193
8.2 Black box fields	194
8.2.1 Functions returning black box fields	195
8.3 Black box algebra	195
8.3.1 Functions returning black box algebras	196
8.4 Black box ring	196
8.5 Black box free Z - <i>p</i> -modules	197
Chapter 9: Operations on general PARI objects	199
9.1 Assignment	199
9.2 Conversions	199

9.2.1 Scalars	199
9.2.2 Modular objects / lifts	201
9.2.3 Between polynomials and coefficient arrays	201
9.3 Constructors	203
9.3.1 Clean constructors	203
9.3.2 Unclean constructors	206
9.3.3 From roots to polynomials	209
9.4 Integer parts	209
9.5 Valuation and shift	210
9.6 Comparison operators	210
9.6.1 Generic	210
9.6.2 Comparison with a small integer	211
9.7 Miscellaneous Boolean functions	212
9.7.1 Obsolete	212
9.8 Sorting	213
9.8.1 Basic sort	213
9.8.2 Indirect sorting	213
9.8.3 Generic sort and search	213
9.8.4 Further useful comparison functions	214
9.9 Divisibility, Euclidean division	215
9.10 GCD, content and primitive part	216
9.10.1 Generic	216
9.10.2 Over the rationals	216
9.11 Generic arithmetic operators	217
9.11.1 Unary operators	217
9.11.2 Binary operators	218
9.12 Generic operators: product, powering, factorback	219
9.13 Matrix and polynomial norms	220
9.14 Substitution and evaluation	221
Chapter 10: Miscellaneous mathematical functions	223
10.1 Fractions	223
10.2 Binomials	223
10.3 Real numbers	223
10.4 Complex numbers	224
10.5 Quadratic numbers and binary quadratic forms	224
10.6 Polynomials	225
10.7 Power series	226
10.8 Functions to handle <code>t_FFELT</code>	226
10.8.1 FFX	229
10.8.2 FFM	230
10.8.3 FFXQ	230
10.9 Transcendental functions	231
10.9.1 Transcendental functions with <code>t_REAL</code> arguments	231
10.9.2 Other complex transcendental functions	232
10.9.3 Modular functions	233
10.9.4 Transcendental functions with <code>t_PADIC</code> arguments	233
10.9.5 Cached constants	233
10.10 Permutations	234
10.11 Small groups	235

Chapter 11: Standard data structures	239
11.1 Character strings	239
11.1.1 Functions returning a <code>char *</code>	239
11.1.2 Functions returning a <code>t_STR</code>	240
11.1.3 Dynamic strings	240
11.2 Output	241
11.2.1 Output contexts	241
11.2.2 Default output context	241
11.2.3 PARI colors	242
11.2.4 Obsolete output functions	242
11.3 Files	243
11.3.1 <code>pariFILE</code>	243
11.3.2 Temporary files	244
11.4 Errors	244
11.4.1 Internal errors, “system” errors	244
11.4.2 Syntax errors, type errors	245
11.4.3 Overflows	246
11.4.4 Errors triggered intentionally	247
11.4.5 Mathematical errors	248
11.4.6 Miscellaneous functions	249
11.5 Hashtables	249
11.6 Dynamic arrays	251
11.6.1 Initialization	251
11.6.2 Adding elements	251
11.6.3 Accessing elements	252
11.6.4 Stack of stacks	252
11.6.5 Public interface	252
11.7 Vectors and Matrices	253
11.7.1 Access and extract	253
11.7.2 Componentwise operations	254
11.7.3 Low-level vectors and columns functions	255
11.8 Vectors of small integers	255
11.8.1 <code>t_VECSMALL</code>	255
11.8.2 Vectors of <code>t_VECSMALL</code>	257
Chapter 12: Functions related to the GP interpreter	259
12.1 Handling closures	259
12.1.1 Functions to evaluate <code>t_CLOSURE</code>	259
12.1.2 Functions to handle control flow changes	260
12.1.3 Functions to deal with lexical local variables	260
12.1.4 Functions returning new closures	260
12.1.5 Functions used by the gp debugger (break loop)	261
12.1.6 Standard wrappers for iterators	261
12.2 Defaults	262
12.3 Records and Lazy vectors	265
Chapter 13: Algebraic Number Theory	267
13.1 General Number Fields	267
13.1.1 Number field types	267
13.1.2 Extracting info from a <code>nf</code> structure	269
13.1.3 Extracting info from a <code>bnf</code> structure	270

13.1.4	Extracting info from a bnr structure	270
13.1.5	Extracting info from an rnf structure	271
13.1.6	Extracting info from a bid structure	271
13.1.7	Extracting info from a znstar structure	272
13.1.8	Inserting info in a number field structure	273
13.1.9	Increasing accuracy	273
13.1.10	Number field arithmetic	274
13.1.11	Elements in factored form	277
13.1.12	Ideal arithmetic	278
13.1.13	Maximal ideals	281
13.1.14	Decomposition group	282
13.1.15	Reducing modulo maximal ideals	282
13.1.16	Valuations	284
13.1.17	Signatures	284
13.1.18	Maximal order and discriminant, conversion to nf structure	285
13.1.19	Computing in the class group	287
13.1.20	Floating point embeddings, the T_2 quadratic form	288
13.1.21	Ideal reduction, low level	289
13.1.22	Ideal reduction, high level	290
13.1.23	Class field theory	291
13.1.24	Grunwald–Wang theorem	292
13.1.25	Relative equations, Galois conjugates	292
13.1.26	Cyclotomics units	294
13.1.27	Obsolete routines	294
13.2	Galois extensions of Q	295
13.2.1	Extracting info from a gal structure	295
13.2.2	Miscellaneous functions	295
13.3	Quadratic number fields and quadratic forms	296
13.3.1	Checks	296
13.3.2	Class number	296
13.3.3	t_QFI , t_QFR	296
13.3.4	Efficient real quadratic forms	298
13.4	Linear algebra over Z	299
13.4.1	Hermite and Smith Normal Forms	299
13.4.2	The LLL algorithm	303
13.4.3	Linear dependencies	304
13.4.4	Reduction modulo matrices	305
13.5	Finite abelian groups and characters	306
13.5.1	Abstract groups	306
13.5.2	Dirichlet characters	307
13.6	Central simple algebras	308
13.6.1	Initialization	308
13.6.2	Type checks	308
13.6.3	Shallow accessors	309
13.6.4	Other low-level functions	309
Chapter 14:	Elliptic curves and arithmetic geometry	311
14.1	Elliptic curves	311
14.1.1	Types of elliptic curves	311
14.1.2	Type checking	311

14.1.3	Extracting info from an <code>ell</code> structure	312
14.1.4	Points	316
14.1.5	Change of variables	316
14.1.6	Generic helper functions	316
14.1.7	Functions to handle elliptic curves over finite fields	317
14.2	Arithmetic on elliptic curve over a finite field in simple form	317
14.2.1	Helper functions	317
14.2.2	Elliptic curves over $\mathbf{F}_p, p > 3$	318
14.2.3	<code>FpE</code>	318
14.2.4	<code>Fle</code>	319
14.2.5	<code>FpJ</code>	320
14.2.6	<code>F1j</code>	320
14.2.7	Elliptic curves over \mathbf{F}_{2^n}	321
14.2.8	<code>F2xqE</code>	321
14.2.9	Elliptic curves over \mathbf{F}_q , small characteristic $p > 2$	322
14.2.10	<code>F1xqE</code>	322
14.2.11	Elliptic curves over \mathbf{F}_q , large characteristic	323
14.2.12	<code>FpXQE</code>	323
14.3	Functions related to modular polynomials	324
14.3.1	Functions related to modular invariants	324
14.4	Other curves	325
Chapter 15:	<i>L</i>-functions	327
15.1	Accessors	327
15.2	Conversions and constructors	328
15.3	Variants of GP functions	328
15.4	Inverse Mellin transforms of Gamma products	329
Chapter 16:	Modular symbols	331
Chapter 17:	Modular forms	331
17.1	Implementation of public data structures	331
17.1.1	Accessors for modular form spaces	331
17.1.2	Accessors for individual modular forms	332
17.1.3	Nebentypus	333
17.1.4	Miscellaneous functions	333
Chapter 18:	Plots	335
18.0.5	Highlevel function	335
18.0.6	Function	335
18.0.7	Obsolete functions	336
18.0.8	Dump rectwindows to a PostScript or SVG file	336
18.0.9	Technical functions exported for convenience	336
Appendix A:	A Sample program and Makefile	337
Appendix B:	PARI and threads	339
Index		342

Chapter 4: Programming PARI in Library Mode

The *User's Guide to Pari/GP* gives in three chapters a general presentation of the system, of the `gp` calculator, and detailed explanation of high level PARI routines available through the calculator. The present manual assumes general familiarity with the contents of these chapters and the basics of ANSI C programming, and focuses on the usage of the PARI library. In this chapter, we introduce the general concepts of PARI programming and describe useful general purpose functions; the following chapters describes all public low or high-level functions, underlying or extending the GP functions seen in Chapter 3 of the User's guide.

4.1 Introduction: initializations, universal objects.

To use PARI in library mode, you must write a C program and link it to the PARI library. See the installation guide or the Appendix to the *User's Guide to Pari/GP* on how to create and install the library and include files. A sample Makefile is presented in Appendix A, and a more elaborate one in `examples/Makefile`. The best way to understand how programming is done is to work through a complete example. We will write such a program in Section 4.10. Before doing this, a few explanations are in order.

First, one must explain to the outside world what kind of objects and routines we are going to use. This is done* with the directive

```
#include <pari/pari.h>
```

In particular, this defines the fundamental type for all PARI objects: the type `GEN`, which is simply a pointer to `long`.

Before any PARI routine is called, one must initialize the system, and in particular the PARI stack which is both a scratchboard and a repository for computed objects. This is done with a call to the function

```
void pari_init(size_t size, ulong maxprime)
```

The first argument is the number of bytes given to PARI to work with, and the second is the upper limit on a precomputed prime number table; `size` should not reasonably be taken below 500000 but you may set `maxprime = 0`, although the system still needs to precompute all primes up to about 2^{16} . For lower-level variants allowing finer control, e.g. preventing PARI from installing its own error or signal handlers, see Section 5.1.2.

We have now at our disposal:

- a PARI *stack* containing nothing. This is a big connected chunk of `size` bytes of memory, where all computations take place. In large computations, intermediate results quickly clutter up memory so some kind of garbage collecting is needed. Most systems do garbage collecting when the memory is getting scarce, and this slows down the performance. PARI takes a different approach,

* This assumes that PARI headers are installed in a directory which belongs to your compiler's search path for header files. You might need to add flags like `-I/usr/local/include` or modify `C_INCLUDE_PATH`.

admittedly more demanding on the programmer: you must do your own cleaning up when the intermediate results are not needed anymore. We will see later how (and when) this is done.

- the following *universal objects* (by definition, objects which do not belong to the stack): the integers 0, 1, -1, 2 and -2 (respectively called `gen_0`, `gen_1`, `gen_m1`, `gen_2` and `gen_m2`), the fraction $\frac{1}{2}$ (`ghalf`). All of these are of type `GEN`.

- a *heap* which is just a linked list of permanent universal objects. For now, it contains exactly the ones listed above. You will probably very rarely use the heap yourself; and if so, only as a collection of copies of objects taken from the stack (called clones in the sequel). Thus you need not bother with its internal structure, which may change as PARI evolves. Some complex PARI functions create clones for special garbage collecting purposes, usually destroying them when returning.

- a table of primes (in fact of *differences* between consecutive primes), called `diffptr`, of type `byteptr` (pointer to `unsigned char`). Its use is described in Section 5.4 later. Using it directly is deprecated, high-level iterators provide a cleaner and more flexible interface, see Section 4.8.2 (such iterators use the private prime table, but extend it dynamically).

- access to all the built-in functions of the PARI library. These are declared to the outside world when you include `pari.h`, but need the above things to function properly. So if you forget the call to `pari_init`, you will get a fatal error when running your program.

4.2 Important technical notes.

4.2.1 Backward compatibility.

The PARI function names evolved over time, and deprecated functions are eventually deleted. The file `pariold.h` contains macros implementing a weak form of backward compatibility. In particular, whenever the name of a documented function changes, a `#define` is added to this file so that the old name expands to the new one (provided the prototype didn't change also).

This file is included by `pari.h`, but a large section is commented out by default. Define `PARI_OLD_NAMES` before including `pari.h` to pollute your namespace with lots of obsolete names like `un*`: that might enable you to compile old programs without having to modify them. The preferred way to do that is to add `-DPARI_OLD_NAMES` to your compiler `CFLAGS`, so that you don't need to modify the program files themselves.

Of course, it's better to fix the program if you can!

4.2.2 Types.

Although PARI objects all have the C type `GEN`, we will freely use the word **type** to refer to PARI dynamic subtypes: `t_INT`, `t_REAL`, etc. The declaration

```
GEN x;
```

declares a C variable of type `GEN`, but its "value" will be said to have type `t_INT`, `t_REAL`, etc. The meaning should always be clear from the context.

* For (long)gen.1. Since 2004 and version 2.2.9, typecasts are completely unnecessary in PARI programs.

4.2.3 Type recursivity.

Conceptually, most PARI types are recursive. But the `GEN` type is a pointer to `long`, not to `GEN`. So special macros must be used to access `GEN`'s components. The simplest one is `gel(V, i)`, where `el` stands for `e`lement, to access component number i of the `GEN` V . This is a valid `lvalue` (may be put on the left side of an assignment), and the following two constructions are exceedingly frequent

```
gel(V, i) = x;  
x = gel(V, i);
```

where x and V are `GEN`s. This macro accesses and modifies directly the components of V and do not create a copy of the coefficient, contrary to all the library *functions*.

More generally, to retrieve the values of elements of lists of ... of lists of vectors we have the `gmael` macros (for `m`ultidimensional `a`rray `e`lement). The syntax is `gmael n (V, a1, ..., a n)`, where V is a `GEN`, the a_i are indexes, and n is an integer between 1 and 5. This stands for $x[a_1][a_2] \dots [a_n]$, and returns a `GEN`. The macros `gel` (resp. `gmael`) are synonyms for `gmael1` (resp. `gmael2`).

Finally, the macro `gcoeff(M, i, j)` has exactly the meaning of $M[i, j]$ in GP when M is a matrix. Note that due to the implementation of `t_MAT`s as horizontal lists of vertical vectors, `gcoeff(x, y)` is actually equivalent to `gmael(y, x)`. One should use `gcoeff` in matrix context, and `gmael` otherwise.

4.2.4 Variations on basic functions. In the library syntax descriptions in Chapter 3, we have only given the basic names of the functions. For example `gadd(x, y)` assumes that x and y are `GEN`s, and *creates* the result $x + y$ on the PARI stack. For most of the basic operators and functions, many other variants are available. We give some examples for `gadd`, but the same is true for all the basic operators, as well as for some simple common functions (a complete list is given in Chapter 6):

```
GEN gaddgs(GEN x, long y)
```

```
GEN gaddsg(long x, GEN y)
```

In the following one, z is a preexisting `GEN` and the result of the corresponding operation is put into z . The size of the PARI stack does not change:

```
void gaddz(GEN x, GEN y, GEN z)
```

(This last form is inefficient in general and deprecated outside of PARI kernel programming.) Low level kernel functions implement these operators for specialized arguments and are also available: Level 0 deals with operations at the word level (`longs` and `ulongs`), Level 1 with `t_INT` and `t_REAL` and Level 2 with the rest (modular arithmetic, polynomial arithmetic and linear algebra). Here are some examples of Level 1 functions:

```
GEN addii(GEN x, GEN y): here  $x$  and  $y$  are GENs of type t_INT (this is not checked).
```

```
GEN addrr(GEN x, GEN y): here  $x$  and  $y$  are GENs of type t_REAL (this is not checked).
```

There also exist functions `addir`, `addri`, `mpadd` (whose two arguments can be of type `t_INT` or `t_REAL`), `addis` (to add a `t_INT` and a `long`) and so on.

The Level 1 names are self-explanatory once you know that `i` stands for a `t_INT`, `r` for a `t_REAL`, `mp` for `i` or `r`, `s` for a signed C long integer, `u` for an unsigned C long integer; finally the suffix `z` means that the result is not created on the PARI stack but assigned to a preexisting `GEN` object passed as an extra argument. Chapter 6 gives a description of these low-level functions.

Level 2 names are more complicated, see Section 7.1 for all the gory details, and we content ourselves with a simple example used to implement `t_INTMOD` arithmetic:

`GEN Fp_add(GEN x, GEN y, GEN m)`: returns the sum of x and y modulo m . Here x, y, m are `t_INTs` (this is not checked). The operation is more efficient if the inputs x, y are reduced modulo m , but this is not a necessary condition.

Important Note. These specialized functions are of course more efficient than the generic ones, but note the hidden danger here: the types of the objects involved (which is not checked) must be severely controlled, e.g. using `addii` on a `t_FRAC` argument will cause disasters. Type mismatches may corrupt the PARI stack, though in most cases they will just immediately overflow the stack. Because of this, the PARI philosophy of giving a result which is as exact as possible, enforced for generic functions like `gadd` or `gmul`, is dropped in kernel routines of Level 1, where it is replaced by the much simpler rule: the result is a `t_INT` if and only if all arguments are integer types (`t_INT` but also `C long` and `ulong`) and a `t_REAL` otherwise. For instance, multiplying a `t_REAL` by a `t_INT` always yields a `t_REAL` if you use `mulir`, where `gmul` returns the `t_INT gen_0` if the integer is 0.

4.2.5 Portability: 32-bit / 64-bit architectures.

PARI supports both 32-bit and 64-bit based machines, but not simultaneously! The library is compiled assuming a given architecture, and some of the header files you include (through `pari.h`) will have been modified to match the library.

Portable macros are defined to bypass most machine dependencies. If you want your programs to run identically on 32-bit and 64-bit machines, you have to use these, and not the corresponding numeric values, whenever the precise size of your `long` integers might matter. Here are the most important ones:

	64-bit	32-bit	
<code>BITS_IN_LONG</code>	64	32	
<code>LONG_IS_64BIT</code>	defined	undefined	
<code>DEFAULTPREC</code>	3	4	(≈ 19 decimal digits, see formula below)
<code>MEDDEFAULTPREC</code>	4	6	(≈ 38 decimal digits)
<code>BIGDEFAULTPREC</code>	5	8	(≈ 57 decimal digits)

For instance, suppose you call a transcendental function, such as

```
GEN gexp(GEN x, long prec).
```

The last argument `prec` is an integer ≥ 3 , corresponding to the default floating point precision required. It is *only* used if `x` is an exact object, otherwise the relative precision is determined by the precision of `x`. Since the parameter `prec` sets the size of the inexact result counted in (`long`) *words* (including codewords), the same value of `prec` will yield different results on 32-bit and 64-bit machines. Real numbers have two codewords (see Section 4.5), so the formula for computing the bit accuracy is

$$\text{bit_accuracy}(\text{prec}) = (\text{prec} - 2) * \text{BITS_IN_LONG}$$

(this is actually the definition of an inline function). The corresponding accuracy expressed in decimal digits would be

$$\text{bit_accuracy}(\text{prec}) * \log(2) / \log(10).$$

For example if the value of `prec` is 5, the corresponding accuracy for 32-bit machines is $(5 - 2) * \log(2^{32}) / \log(10) \approx 28$ decimal digits, while for 64-bit machines it is $(5 - 2) * \log(2^{64}) / \log(10) \approx 57$ decimal digits.

Thus, you must take care to change the `prec` parameter you are supplying according to the bit size, either using the default precisions given by the various `DEFAULTPRECS`, or by using conditional constructs of the form:

```
#ifndef LONG_IS_64BIT
    prec = 4;
#else
    prec = 6;
#endif
```

which is in this case equivalent to the statement `prec = MEDDEFAULTPREC;`.

Note that for parity reasons, half the accuracies available on 32-bit architectures (the odd ones) have no precise equivalents on 64-bit machines.

4.2.6 Using `malloc` / `free`. You should make use of the PARI stack as much as possible, and avoid allocating objects using the customary functions. If you do, you should use, or at least have a very close look at, the following wrappers:

`void* pari_malloc(size_t size)` calls `malloc` to allocate `size` bytes and returns a pointer to the allocated memory. If the request fails, an error is raised. The `SIGINT` signal is blocked until `malloc` returns, to avoid leaving the system stack in an inconsistent state.

`void* pari_realloc(void* ptr, size_t size)` as `pari_malloc` but calls `realloc` instead of `malloc`.

`void* pari_calloc(size_t size)` as `pari_malloc`, setting the memory to zero.

`void pari_free(void* ptr)` calls `free` to liberate the memory space pointed to by `ptr`, which must have been allocated by `malloc` (`pari_malloc`) or `realloc` (`pari_realloc`). The `SIGINT` signal is blocked until `free` returns.

If you use the standard `libc` functions instead of our wrappers, then your functions will be subtly incompatible with the `gp` calculator: when the user tries to interrupt a computation, the calculator may crash (if a system call is interrupted at the wrong time).

4.3 Garbage collection.

4.3.1 Why and how.

As we have seen, `pari_init` allocates a big range of addresses, the *stack*, that are going to be used throughout. Recall that all PARI objects are pointers. Except for a few universal objects, they all point at some part of the stack.

The stack starts at the address `bot` and ends just before `top`. This means that the quantity

$$(\text{top} - \text{bot}) / \text{sizeof}(\text{long})$$

is (roughly) equal to the `size` argument of `pari_init`. The PARI stack also has a “current stack pointer” called `avma`, which stands for **available memory address**. These three variables are global (declared by `pari.h`). They are of type `pari_sp`, which means *pari stack pointer*.

The stack is oriented upside-down: the more recent an object, the closer to `bot`. Accordingly, initially `avma = top`, and `avma` gets *decremented* as new objects are created. As its name indicates,

`avma` always points just *after* the first free address on the stack, and `(GEN)avma` is always (a pointer to) the latest created object. When `avma` reaches `bot`, the stack overflows, aborting all computations, and an error message is issued. To avoid this *you* need to clean up the stack from time to time, when intermediate objects are not needed anymore. This is called “*garbage collecting.*”

We are now going to describe briefly how this is done. We will see many concrete examples in the next subsection.

- First, PARI routines do their own garbage collecting, which means that whenever a documented function from the library returns, only its result(s) have been added to the stack, possibly up to a very small overhead (non-documented ones may not do this). In particular, a PARI function that does not return a `GEN` does not clutter the stack. Thus, if your computation is small enough (e.g. you call few PARI routines, or most of them return `long` integers), then you do not need to do any garbage collecting. This is probably the case in many of your subroutines. Of course the objects that were on the stack *before* the function call are left alone. Except for the ones listed below, PARI functions only collect their own garbage.

- It may happen that all objects that were created after a certain point can be deleted — for instance, if the final result you need is not a `GEN`, or if some search proved futile. Then, it is enough to record the value of `avma` just *before* the first garbage is created, and restore it upon exit:

```
pari_sp av = avma; /* record initial avma */
garbage ...
avma = av; /* restore it */
```

All objects created in the `garbage` zone will eventually be overwritten: they should no longer be accessed after `avma` has been restored.

- If you want to destroy (i.e. give back the memory occupied by) the *latest* PARI object on the stack (e.g. the latest one obtained from a function call), you can use the function

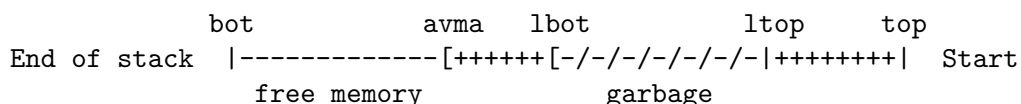
```
void cgiv(GEN z)
```

where `z` is the object you want to give back. This is equivalent to the above where the initial `av` is computed from `z`.

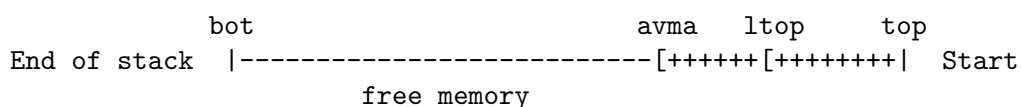
- Unfortunately life is not so simple, and sometimes you will want to give back accumulated garbage *during* a computation without losing recent data. We shall start with the lowest level function to get a feel for the underlying mechanisms, we shall describe simpler variants later:

`GEN gerepile(pari_sp ltop, pari_sp lbot, GEN q)`. This function cleans up the stack between `ltop` and `lbot`, where `lbot < ltop`, and returns the updated object `q`. This means:

1) we translate (copy) all the objects in the interval `[avma, lbot[`, so that its right extremity abuts the address `ltop`. Graphically



becomes:



where ++ denote significant objects, -- the unused part of the stack, and -/- the garbage we remove.

2) The function then inspects all the PARI objects between `avma` and `lbot` (i.e. the ones that we want to keep and that have been translated) and looks at every component of such an object which is not a codeword. Each such component is a pointer to an object whose address is either

- between `avma` and `lbot`, in which case it is suitably updated,
- larger than or equal to `ltop`, in which case it does not change, or
- between `lbot` and `ltop` in which case `gerepile` raises an error (“significant pointers lost in `gerepile`”).

3) `avma` is updated (we add `ltop - lbot` to the old value).

4) We return the (possibly updated) object `q`: if `q` initially pointed between `avma` and `lbot`, we return the updated address, as in 2). If not, the original address is still valid, and is returned!

As stated above, no component of the remaining objects (in particular `q`) should belong to the erased segment `[lbot, ltop[`, and this is checked within `gerepile`. But beware as well that the addresses of the objects in the translated zone change after a call to `gerepile`, so you must not access any pointer which previously pointed into the zone below `ltop`. If you need to recover more than one object, use the `gerepileall` function below.

Remark. As a consequence of the preceding explanation, if a PARI object is to be relocated by `gerepile` then, apart from universal objects, the chunks of memory used by its components should be in consecutive memory locations. All GENS created by documented PARI functions are guaranteed to satisfy this. This is because the `gerepile` function knows only about *two connected zones*: the garbage that is erased (between `lbot` and `ltop`) and the significant pointers that are copied and updated. If there is garbage interspersed with your objects, disaster occurs when we try to update them and consider the corresponding “pointers”. In most cases of course the said garbage is in fact a bunch of other GENS, in which case we simply waste time copying and updating them for nothing. But be wary when you allow objects to become disconnected.

In practice this is achieved by the following programming idiom:

```
ltop = avma; garbage(); lbot = avma; q = anything();
return gerepile(ltop, lbot, q); /* returns the updated q */
```

or directly

```
ltop = avma; garbage(); lbot = avma;
return gerepile(ltop, lbot, anything());
```

Beware that

```
ltop = avma; garbage();
return gerepile(ltop, avma, anything())
```

might work, but should be frowned upon. We cannot predict whether `avma` is evaluated after or before the call to `anything()`: it depends on the compiler. If we are out of luck, it is *after* the call, so the result belongs to the garbage zone and the `gerepile` statement becomes equivalent to `avma = ltop`. Thus we return a pointer to random garbage.

4.3.2 Variants.

GEN `gerepileupto(pari_sp ltop, GEN q)`. Cleans the stack between `ltop` and the *connected* object `q` and returns `q` updated. For this to work, `q` must have been created *before* all its components, otherwise they would belong to the garbage zone! Unless mentioned otherwise, documented PARI functions guarantee this.

GEN `gerepilecopy(pari_sp ltop, GEN x)`. Functionally equivalent to, but more efficient than
`gerepileupto(ltop, gcopy(x))`

In this case, the GEN parameter `x` need not satisfy any property before the garbage collection: it may be disconnected, components created before the root, and so on. Of course, this is about twice slower than either `gerepileupto` or `gerepile`, because `x` has to be copied to a clean stack zone first. This function is a special case of `gerepileall` below, where $n = 1$.

`void gerepileall(pari_sp ltop, int n, ...)`. To cope with complicated cases where many objects have to be preserved. The routine expects n further arguments, which are the *addresses* of the GENs you want to preserve:

```
pari_sp ltop = avma;
...; y = ...; ... x = ...; ...;
gerepileall(ltop, 2, &x, &y);
```

It cleans up the most recent part of the stack (between `ltop` and `avma`), updating all the GENs added to the argument list. A copy is done just before the cleaning to preserve them, so they do not need to be connected before the call. With `gerepilecopy`, this is the most robust of the `gerepile` functions (the less prone to user error), hence the slowest.

`void gerepileallsp(pari_sp ltop, pari_sp lbot, int n, ...)`. More efficient, but trickier than `gerepileall`. Cleans the stack between `lbot` and `ltop` and updates the GENs pointed at by the elements of `gptr` without any further copying. This is subject to the same restrictions as `gerepile`, the only difference being that more than one address gets updated.

4.3.3 Examples.

4.3.3.1 `gerepile`.

Let `x` and `y` be two preexisting PARI objects and suppose that we want to compute $x^2 + y^2$. This is done using the following program:

```
GEN x2 = gsqr(x);
GEN y2 = gsqr(y), z = gadd(x2,y2);
```

The GEN `z` indeed points at the desired quantity. However, consider the stack: it contains as unnecessary garbage `x2` and `y2`. More precisely it contains (in this order) `z`, `y2`, `x2`. (Recall that, since the stack grows downward from the top, the most recent object comes first.)

It is not possible to get rid of `x2`, `y2` before `z` is computed, since they are used in the final operation. We cannot record `avma` before `x2` is computed and restore it later, since this would destroy `z` as well. It is not possible either to use the function `cgiv` since `x2` and `y2` are not at the bottom of the stack and we do not want to give back `z`.

But using `gerepile`, we can give back the memory locations corresponding to `x2`, `y2`, and move the object `z` upwards so that no space is lost. Specifically:

```
pari_sp ltop = avma; /* remember the current top of the stack */
```

```

GEN x2 = gsqr(x);
GEN y2 = gsqr(y);
pari_sp lbot = avma; /* the bottom of the garbage pile */
GEN z = gadd(x2, y2); /* z is now the last object on the stack */
z = gerepile(ltop, lbot, z);

```

Of course, the last two instructions could also have been written more simply:

```
z = gerepile(ltop, lbot, gadd(x2,y2));
```

In fact `gerepileupto` is even simpler to use, because the result of `gadd` is the last object on the stack and `gadd` is guaranteed to return an object suitable for `gerepileupto`:

```
ltop = avma;
z = gerepileupto(ltop, gadd(gsqr(x), gsqr(y)));
```

Make sure you understand exactly what has happened before you go on!

Remark on assignments and `gerepile`. When the tree structure and the size of the PARI objects which will appear in a computation are under control, one may allocate sufficiently large objects at the beginning, use assignment statements, then simply restore `avma`. Coming back to the above example, note that *if* we know that `x` and `y` are of type real fitting into `DEFAULTPREC` words, we can program without using `gerepile` at all:

```
z = cgetr(DEFAULTPREC); ltop = avma;
gaffect(gadd(gsqr(x), gsqr(y)), z);
avma = ltop;
```

This is often *slower* than a craftily used `gerepile` though, and certainly more cumbersome to use. As a rule, assignment statements should generally be avoided.

Variations on a theme. it is often necessary to do several `gerepiles` during a computation. However, the fewer the better. The only condition for `gerepile` to work is that the garbage be connected. If the computation can be arranged so that there is a minimal number of connected pieces of garbage, then it should be done that way.

For example suppose we want to write a function of two `GEN` variables `x` and `y` which creates the vector $[x^2 + y, y^2 + x]$. Without garbage collecting, one would write:

```
p1 = gsqr(x); p2 = gadd(p1, y);
p3 = gsqr(y); p4 = gadd(p3, x);
z = mkvec2(p2, p4); /* not suitable for gerepileupto! */
```

This leaves a dirty stack containing (in this order) `z`, `p4`, `p3`, `p2`, `p1`. The garbage here consists of `p1` and `p3`, which are separated by `p2`. But if we compute `p3` *before* `p2` then the garbage becomes connected, and we get the following program with garbage collecting:

```
ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(p1,y);
gel(z, 2) = gadd(p3,x); z = gerepile(ltop,lbot,z);
```

Finishing by `z = gerepileupto(ltop, z)` would be ok as well. Beware that

```
ltop = avma; p1 = gadd(gsqr(x), y); p3 = gadd(gsqr(y), x);
z = cgetg(3, t_VEC);
```

```

    gel(z, 1) = p1;
    gel(z, 2) = p3; z = gerepileupto(ltop,z); /* WRONG */

```

is a disaster since `p1` and `p3` are created before `z`, so the call to `gerepileupto` overwrites them, leaving `gel(z, 1)` and `gel(z, 2)` pointing at random data! The following does work:

```

    ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
    lbot = avma; z = mkvec2(gadd(p1,y), gadd(p3,x));
    z = gerepile(ltop,lbot,z);

```

but is very subtly wrong in the sense that `z = gerepileupto(ltop, z)` would *not* work. The reason being that `mkvec2` creates the root `z` of the vector *after* its arguments have been evaluated, creating the components of `z` too early; `gerepile` does not care, but the created `z` is a time bomb which will explode on any later `gerepileupto`. On the other hand

```

    ltop = avma; z = cgetg(3, t_VEC);
    gel(z, 1) = gadd(gsqr(x), y);
    gel(z, 2) = gadd(gsqr(y), x); z = gerepileupto(ltop,z); /* INEFFICIENT */

```

leaves the results of `gsqr(x)` and `gsqr(y)` on the stack (and lets `gerepileupto` update them for naught). Finally, the most elegant and efficient version (with respect to time and memory use) is as follows

```

    z = cgetg(3, t_VEC);
    ltop = avma; gel(z, 1) = gerepileupto(ltop, gadd(gsqr(x), y));
    ltop = avma; gel(z, 2) = gerepileupto(ltop, gadd(gsqr(y), x));

```

which avoids updating the container `z` and cleans up its components individually, as soon as they are computed.

One last example. Let us compute the product of two complex numbers x and y , using the $3M$ method which requires 3 multiplications instead of the obvious 4. Let $z = x*y$, and set $x = x_r + i*x_i$ and similarly for y and z . We compute $p_1 = x_r * y_r$, $p_2 = x_i * y_i$, $p_3 = (x_r + x_i) * (y_r + y_i)$, and then we have $z_r = p_1 - p_2$, $z_i = p_3 - (p_1 + p_2)$. The program is as follows:

```

    ltop = avma;
    p1 = gmul(gel(x,1), gel(y,1));
    p2 = gmul(gel(x,2), gel(y,2));
    p3 = gmul(gadd(gel(x,1), gel(x,2)), gadd(gel(y,1), gel(y,2)));
    p4 = gadd(p1,p2);
    lbot = avma; z = cgetg(3, t_COMPLEX);
    gel(z, 1) = gsub(p1,p2);
    gel(z, 2) = gsub(p3,p4); z = gerepile(ltop,lbot,z);

```

Exercise. Write a function which multiplies a matrix by a column vector. Hint: start with a `cgetg` of the result, and use `gerepile` whenever a coefficient of the result vector is computed. You can look at the answer in `src/basemath/RgV.c:RgM_RgC_mul()`.

4.3.3.2 `gerepileall`.

Let us now see why we may need the `gerepileall` variants. Although it is not an infrequent occurrence, we do not give a specific example but a general one: suppose that we want to do a computation (usually inside a larger function) producing more than one PARI object as a result, say two for instance. Then even if we set up the work properly, before cleaning up we have a stack which has the desired results `z1`, `z2` (say), and then connected garbage from `lbot` to `ltop`. If we write

```
z1 = gerepile(ltop, lbot, z1);
```

then the stack is cleaned, the pointers fixed up, but we have lost the address of `z2`. This is where we need the `gerepileall` function:

```
gerepileall(ltop, 2, &z1, &z2)
```

copies `z1` and `z2` to new locations, cleans the stack from `ltop` to the old `avma`, and updates the pointers `z1` and `z2`. Here we do not assume anything about the stack: the garbage can be disconnected and `z1`, `z2` need not be at the bottom of the stack. If all of these assumptions are in fact satisfied, then we can call `gerepilemanysp` instead, which is usually faster since we do not need the initial copy (on the other hand, it is less cache friendly).

A most important usage is “random” garbage collection during loops whose size requirements we cannot (or do not bother to) control in advance:

```
pari_sp av = avma;
GEN x, y;
while (...)
{
  garbage(); x = anything();
  garbage(); y = anything(); garbage();
  if (gc_needed(av,1)) /* memory is running low (half spent since entry) */
    gerepileall(av, 2, &x, &y);
}
```

Here we assume that only `x` and `y` are needed from one iteration to the next. As it would be costly to call `gerepile` once for each iteration, we only do it when it seems to have become necessary.

More precisely, the macro `stack_lim(av,n)` denotes an address where $2^{n-1}/(2^{n-1} + 1)$ of the remaining stack space since reference point `av` is exhausted (1/2 for $n = 1$, 2/3 for $n = 2$). The test `gc_needed(av,n)` becomes true whenever `avma` drops below that address.

4.3.4 Comments.

First, `gerepile` has turned out to be a flexible and fast garbage collector for number-theoretic computations, which compares favorably with more sophisticated methods used in other systems. Our benchmarks indicate that the price paid for using `gerepile` and `gerepile`-related copies, when properly used, is usually less than 1% of the total running time, which is quite acceptable!

Second, it is of course harder on the programmer, and quite error-prone if you do not stick to a consistent PARI programming style. If all seems lost, just use `gerepilecopy` (or `gerepileall`) to fix up the stack for you. You can always optimize later when you have sorted out exactly which routines are crucial and what objects need to be preserved and their usual sizes.

If you followed us this far, congratulations, and rejoice: the rest is much easier.

4.4 Creation of PARI objects, assignments, conversions.

4.4.1 Creation of PARI objects. The basic function which creates a PARI object is

`GEN cgetg(long l, long t)` l specifies the number of longwords to be allocated to the object, and t is the type of the object, in symbolic form (see Section 4.5 for the list of these). The precise effect of this function is as follows: it first creates on the PARI *stack* a chunk of memory of size `length` longwords, and saves the address of the chunk which it will in the end return. If the stack has been used up, a message to the effect that “the PARI stack overflows” is printed, and an error raised. Otherwise, it sets the type and length of the PARI object. In effect, it fills its first codeword (`z[0]`). Many PARI objects also have a second codeword (types `t_INT`, `t_REAL`, `t_PADIC`, `t_POL`, and `t_SER`). In case you want to produce one of those from scratch, which should be exceedingly rare, *it is your responsibility to fill this second codeword*, either explicitly (using the macros described in Section 4.5), or implicitly using an assignment statement (using `gaffect`).

Note that the length argument l is predetermined for a number of types: 3 for types `t_INTMOD`, `t_FRAC`, `t_COMPLEX`, `t_POLMOD`, `t_RFRAC`, 4 for type `t_QUAD` and `t_QFI`, and 5 for type `t_PADIC` and `t_QFR`. However for the sake of efficiency, `cgetg` does not check this: disasters will occur if you give an incorrect length for those types.

Notes. 1) The main use of this function is create efficiently a constant object, or to prepare for later assignments (see Section 4.4.3). Most of the time you will use `GEN` objects as they are created and returned by PARI functions. In this case you do not need to use `cgetg` to create space to hold them.

2) For the creation of leaves, i.e. `t_INT` or `t_REAL`,

```
GEN cgeti(long length)
```

```
GEN cgetr(long length)
```

should be used instead of `cgetg(length, t_INT)` and `cgetg(length, t_REAL)` respectively. Finally

```
GEN cgetc(long prec)
```

creates a `t_COMPLEX` whose real and imaginary part are `t_REALs` allocated by `cgetr(prec)`.

Examples. 1) Both `z = cgeti(DEFAULTPREC)` and `cgetg(DEFAULTPREC, t_INT)` create a `t_INT` whose “precision” is `bit_accuracy(DEFAULTPREC) = 64`. This means `z` can hold rational integers of absolute value less than 2^{64} . Note that in both cases, the second codeword is *not* filled. Of course we could use numerical values, e.g. `cgeti(4)`, but this would have different meanings on different machines as `bit_accuracy(4)` equals 64 on 32-bit machines, but 128 on 64-bit machines.

2) The following creates a *complex number* whose real and imaginary parts can hold real numbers of precision `bit_accuracy(MEDDEFAULTPREC) = 96` bits:

```
z = cgetg(3, t_COMPLEX);
gel(z, 1) = cgetr(MEDDEFAULTPREC);
gel(z, 2) = cgetr(MEDDEFAULTPREC);
```

or simply `z = cgetc(MEDDEFAULTPREC)`.

3) To create a matrix object for 4×3 matrices:

```
z = cgetg(4, t_MAT);
for(i=1; i<4; i++) gel(z, i) = cgetg(5, t_COL);
```

or simply `z = zeromatcopy(4, 3)`, which further initializes all entries to `gen_0`.

These last two examples illustrate the fact that since PARI types are recursive, all the branches of the tree must be created. The function `cgetg` creates only the “root”, and other calls to `cgetg` must be made to produce the whole tree. For matrices, a common mistake is to think that `z = cgetg(4, t_MAT)` (for example) creates the root of the matrix: one needs also to create the column vectors of the matrix (obviously, since we specified only one dimension in the first `cgetg`!). This is because a matrix is really just a row vector of column vectors (hence a priori not a basic type), but it has been given a special type number so that operations with matrices become possible.

Finally, to facilitate input of constant objects when speed is not paramount, there are four `varargs` functions:

`GEN mkintn(long n, ...)` returns the non-negative `t_INT` whose development in base 2^{32} is given by the following n 32bit-words (`unsigned int`).

```
mkintn(3, a2, a1, a0);
```

returns $a_2 2^{64} + a_1 2^{32} + a_0$.

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose n coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose n coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose n coefficients (`GEN`) follow.

Warning. Contrary to the policy of general PARI functions, the latter three functions do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. For instance

```
/* gerepile-safe: components are universal objects */
z = mkvecn(3, gen_1, gen_0, gen_2);
/* not OK for gerepileupto: stoi(3) creates component before root */
z = mkvecn(3, stoi(3), gen_0, gen_2);
/* NO! First vector component x is destroyed */
x = gclone(gen_1);
z = mkvecn(3, x, gen_0, gen_2);
gclone(x);
```

The following function is also available as a special case of `mkintn`:

```
GEN uu32toi(ulong a, ulong b)
```

Returns the GEN equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

4.4.2 Sizes.

`long gsizeword(GEN x)` returns the total number of BITS_IN_LONG-bit words occupied by the tree representing `x`.

`long gsizebyte(GEN x)` returns the total number of bytes occupied by the tree representing `x`, i.e. `gsizeword(x)` multiplied by `sizeof(long)`. This is normally useless since PARI functions use a number of *words* as input for lengths and precisions.

4.4.3 Assignments. Firstly, if `x` and `y` are both declared as GEN (i.e. pointers to something), the ordinary C assignment `y = x` makes perfect sense: we are just moving a pointer around. However, physically modifying either `x` or `y` (for instance, `x[1] = 0`) also changes the other one, which is usually not desirable.

Very important note. Using the functions described in this paragraph is inefficient and often awkward: one of the `gerepile` functions (see Section 4.3) should be preferred. See the paragraph end for one exception to this rule.

The general PARI assignment function is the function `gaffect` with the following syntax:

```
void gaffect(GEN x, GEN y)
```

Its effect is to assign the PARI object `x` into the *preexisting* object `y`. Both `x` and `y` must be *scalar* types. For convenience, vector or matrices of scalar types are also allowed.

This copies the whole structure of `x` into `y` so many conditions must be met for the assignment to be possible. For instance it is allowed to assign a `t_INT` into a `t_REAL`, but the converse is forbidden. For that, you must use the truncation or rounding function of your choice, e.g. `mpfloor`.

It can also happen that `y` is not large enough or does not have the proper tree structure to receive the object `x`. For instance, let `y` the zero integer with length equal to 2; then `y` is too small to accommodate any non-zero `t_INT`. In general common sense tells you what is possible, keeping in mind the PARI philosophy which says that if it makes sense it is valid. For instance, the assignment of an imprecise object into a precise one does *not* make sense. However, a change in precision of imprecise objects is allowed, even if it *increases* its accuracy: we complement the

“mantissa” with infinitely many 0 digits in this case. (Mantissa between quotes, because this is not restricted to `t_REALs`, it also applies for p -adics for instance.)

All functions ending in “z” such as `gaddz` (see Section 4.2.4) implicitly use this function. In fact what they exactly do is record `avma` (see Section 4.3), perform the required operation, `gaffect` the result to the last operand, then restore the initial `avma`.

You can assign ordinary C long integers into a PARI object (not necessarily of type `t_INT`) using

```
void gaffsg(long s, GEN y)
```

Note. Due to the requirements mentioned above, it is usually a bad idea to use `gaffect` statements. There is one exception: for simple objects (e.g. leaves) whose size is controlled, they can be easier to use than `gerepile`, and about as efficient.

Coercion. It is often useful to coerce an inexact object to a given precision. For instance at the beginning of a routine where precision can be kept to a minimum; otherwise the precision of the input is used in all subsequent computations, which is inefficient if the latter is known to thousands of digits. One may use the `gaffect` function for this, but it is easier and more efficient to call

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of length `prec`.

4.4.4 Copy. It is also very useful to copy a PARI object, not just by moving around a pointer as in the `y = x` example, but by creating a copy of the whole tree structure, without pre-allocating a possibly complicated `y` to use with `gaffect`. The function which does this is called `gcopy`. Its syntax is:

```
GEN gcopy(GEN x)
```

and the effect is to create a new copy of `x` on the PARI stack.

Sometimes, on the contrary, a quick copy of the skeleton of `x` is enough, leaving pointers to the original data in `x` for the sake of speed instead of making a full recursive copy. Use `GEN shallowcopy(GEN x)` for this. Note that the result is not suitable for `gerepileupto` !

Make sure at this point that you understand the difference between `y = x`, `y = gcopy(x)`, `y = shallowcopy(x)` and `gaffect(x,y)`.

4.4.5 Clones. Sometimes, it is more efficient to create a *persistent* copy of a PARI object. This is not created on the stack but on the heap, hence unaffected by `gerepile` and friends. The function which does this is called `gclone`. Its syntax is:

```
GEN gclone(GEN x)
```

A clone can be removed from the heap (thus destroyed) using

```
void gunclone(GEN x)
```

No PARI object should keep references to a clone which has been destroyed!

4.4.6 Conversions. The following functions convert C objects to PARI objects (creating them on the stack as usual):

GEN `stoi(long s)`: C long integer (“small”) to `t_INT`.

GEN `dbltor(double s)`: C double to `t_REAL`. The accuracy of the result is 19 decimal digits, i.e. a type `t_REAL` of length `DEFAULTPREC`, although on 32-bit machines only 16 of them are significant.

We also have the converse functions:

long `itos(GEN x)`: `x` must be of type `t_INT`,

double `rtodbl(GEN x)`: `x` must be of type `t_REAL`,

as well as the more general ones:

long `gtolong(GEN x)`,

double `gtodouble(GEN x)`.

4.5 Implementation of the PARI types.

We now go through each type and explain its implementation. Let `z` be a `GEN`, pointing at a PARI object. In the following paragraphs, we will constantly mix two points of view: on the one hand, `z` is treated as the C pointer it is, on the other, as PARI’s handle on some mathematical entity, so we will shamelessly write `z ≠ 0` to indicate that the *value* thus represented is nonzero (in which case the *pointer* `z` is certainly non-NULL). We offer no apologies for this style. In fact, you had better feel comfortable juggling both views simultaneously in your mind if you want to write correct PARI programs.

Common to all the types is the first codeword `z[0]`, which we do not have to worry about since this is taken care of by `cgetg`. Its precise structure depends on the machine you are using, but it always contains the following data: the *internal type number* attached to the symbolic type name, the *length* of the root in longwords, and a technical bit which indicates whether the object is a clone or not (see Section 4.4.5). This last one is used by `gp` for internal garbage collecting, you will not have to worry about it.

Some types have a second codeword, different for each type, which we will soon describe as we will shortly consider each of them in turn.

The first codeword is handled through the following *macros*:

long `typ(GEN z)` returns the type number of `z`.

void `settyp(GEN z, long n)` sets the type number of `z` to `n` (you should not have to use this function if you use `cgetg`).

long `lg(GEN z)` returns the length (in longwords) of the root of `z`.

long `setlg(GEN z, long l)` sets the length of `z` to `l`; you should not have to use this function if you use `cgetg`.

void `lg_increase(GEN z)` increase the length of `z` by 1; you should not have to use this function if you use `cgetg`.

long `isclone(GEN z)` is `z` a clone?

void `setisclone(GEN z)` sets the *clone* bit.

void `unsetisclone(GEN z)` clears the *clone* bit.

Important remark. For the sake of efficiency, none of the codeword-handling macros check the types of their arguments even when there are stringent restrictions on their use. It is trivial to create invalid objects, or corrupt one of the “universal constants” (e.g. setting the sign of `gen_0` to 1), and they usually provide negligible savings. Use higher level functions whenever possible.

Remark. The clone bit is there so that `gunclone` can check it is deleting an object which was allocated by `gclone`. Miscellaneous vector entries are often cloned by `gp` so that a GP statement like `v[1] = x` does not involve copying the whole of `v`: the component `v[1]` is deleted if its clone bit is set, and is replaced by a clone of `x`. Don't set/unset yourself the clone bit unless you know what you are doing: in particular *never* set the clone bit of a vector component when the said vector is scheduled to be uncloned. Hackish code may abuse the clone bit to tag objects for reasons unrelated to the above instead of using proper data structures. Don't do that.

4.5.1 Type `t_INT` (integer). this type has a second codeword `z[1]` which contains the following information:

the sign of `z`: coded as 1, 0 or -1 if $z > 0$, $z = 0$, $z < 0$ respectively.

the *effective length* of `z`, i.e. the total number of significant longwords. This means the following: apart from the integer 0, every integer is “normalized”, meaning that the most significant mantissa longword is non-zero. However, the integer may have been created with a longer length. Hence the “length” which is in `z[0]` can be larger than the “effective length” which is in `z[1]`.

This information is handled using the following macros:

`long signe(GEN z)` returns the sign of `z`.

`void setsigne(GEN z, long s)` sets the sign of `z` to `s`.

`long lgefint(GEN z)` returns the effective length of `z`.

`void setlgefint(GEN z, long l)` sets the effective length of `z` to `l`.

The integer 0 can be recognized either by its sign being 0, or by its effective length being equal to 2. Now assume that $z \neq 0$, and let

$$|z| = \sum_{i=0}^n z_i B^i, \quad \text{where } z_n \neq 0 \text{ and } B = 2^{\text{BITS_IN_LONG}}.$$

With these notations, n is `lgefint(z) - 3`, and the mantissa of `z` may be manipulated via the following interface:

`GEN int_MSW(GEN z)` returns a pointer to the most significant word of `z`, z_n .

`GEN int_LSW(GEN z)` returns a pointer to the least significant word of `z`, z_0 .

`GEN int_W(GEN z, long i)` returns the i -th significant word of `z`, z_i . Accessing the i -th significant word for $i > n$ yields unpredictable results.

`GEN int_W_lg(GEN z, long i, long lz)` returns the i -th significant word of `z`, z_i , assuming `lgefint(z)` is `lz` ($= n + 3$). Accessing the i -th significant word for $i > n$ yields unpredictable results.

`GEN int_precW(GEN z)` returns the previous (less significant) word of `z`, z_{i-1} assuming `z` points to z_i .

`GEN int_nextW(GEN z)` returns the next (more significant) word of z , z_{i+1} assuming z points to z_i .

Unnormalized integers, such that z_n is possibly 0, are explicitly forbidden. To enforce this, one may write an arbitrary mantissa then call

```
void int_normalize(GEN z, long known0)
```

normalizes in place a non-negative integer (such that z_n is possibly 0), assuming at least the first `known0` words are zero.

For instance a binary `and` could be implemented in the following way:

```
GEN AND(GEN x, GEN y) {
    long i, lx, ly, lout;
    long *xp, *yp, *outp; /* mantissa pointers */
    GEN out;

    if (!signe(x) || !signe(y)) return gen_0;
    lx = lgefint(x); xp = int_LSW(x);
    ly = lgefint(y); yp = int_LSW(y); lout = min(lx,ly); /* > 2 */
    out = cgeti(lout); out[1] = evalsigne(1) | evallgefint(lout);
    outp = int_LSW(out);
    for (i=2; i < lout; i++)
    {
        *outp = (*xp) & (*yp);
        outp = int_nextW(outp);
        xp = int_nextW(xp);
        yp = int_nextW(yp);
    }
    if ( !*int_MSW(out) ) out = int_normalize(out, 1);
    return out;
}
```

This low-level interface is mandatory in order to write portable code since PARI can be compiled using various multiprecision kernels, for instance the native one or GNU MP, with incompatible internal structures (for one thing, the mantissa is oriented in different directions).

4.5.2 Type `t_REAL` (real number). this type has a second codeword $z[1]$ which also encodes its sign, obtained or set using the same functions as for a `t_INT`, and a binary exponent. This exponent is handled using the following macros:

`long expo(GEN z)` returns the exponent of z . This is defined even when z is equal to zero.

`void setexpo(GEN z, long e)` sets the exponent of z to e .

Note the functions:

`long gexpo(GEN z)` which tries to return an exponent for z , even if z is not a real number.

`long gsigne(GEN z)` which returns a sign for z , even when z is neither real nor integer (a rational number for instance).

The real zero is characterized by having its sign equal to 0. If z is not equal to 0, then it is represented as $2^e M$, where e is the exponent, and $M \in [1, 2[$ is the mantissa of z , whose digits are stored in $z[2], \dots, z[\lg(z) - 1]$.

More precisely, let m be the integer $(z[2], \dots, z[\lg(z)-1])$ in base $2^{\text{BITS_IN_LONG}}$; here, $z[2]$ is the most significant longword and is normalized, i.e. its most significant bit is 1. Then we have $M := m/2^{\text{bit_accuracy}(\lg(z))-1-\text{expo}(z)}$.

`GEN mantissa_real(GEN z, long *e)` returns the mantissa m of z , and sets $*e$ to the exponent $\text{bit_accuracy}(\lg(z)) - 1 - \text{expo}(z)$, so that $z = m/2^e$.

Thus, the real number 3.5 to accuracy $\text{bit_accuracy}(\lg(z))$ is represented as $z[0]$ (encoding `type = t_REAL`, $\lg(z)$), $z[1]$ (encoding `sign = 1`, `expo = 1`), $z[2] = 0xe0000000$, $z[3] = \dots = z[\lg(z) - 1] = 0x0$.

4.5.3 Type `t_INTMOD`. $z[1]$ points to the modulus, and $z[2]$ at the number representing the class z . Both are separate GEN objects, and both must be `t_INTs`, satisfying the inequality $0 \leq z[2] < z[1]$.

4.5.4 Type `t_FRAC` (rational number). $z[1]$ points to the numerator n , and $z[2]$ to the denominator d . Both must be of type `t_INT` such that $n \neq 0$, $d > 0$ and $(n, d) = 1$.

4.5.5 Type `t_FFELT` (finite field element). (Experimental)

Components of this type should normally not be accessed directly. Instead, finite field elements should be created using `ffgen`.

The second codeword $z[1]$ determines the storage format of the element, among

- `t_FF_FpXq`: $A=z[2]$ and $T=z[3]$ are `FpX`, $p=z[4]$ is a `t_INT`, where p is a prime number, T is irreducible modulo p , and $\deg A < \deg T$. This represents the element $A \pmod{T}$ in $\mathbf{F}_p[X]/T$.

- `t_FF_Flxq`: $A=z[2]$ and $T=z[3]$ are `Flx`, $l=z[4]$ is a `t_INT`, where l is a prime number, T is irreducible modulo l , and $\deg A < \deg T$. This represents the element $A \pmod{T}$ in $\mathbf{F}_l[X]/T$.

- `t_FF_F2xq`: $A=z[2]$ and $T=z[3]$ are `F2x`, $l=z[4]$ is the `t_INT` 2, T is irreducible modulo 2, and $\deg A < \deg T$. This represents the element $A \pmod{T}$ in $\mathbf{F}_2[X]/T$.

4.5.6 Type `t_COMPLEX` (complex number). $z[1]$ points to the real part, and $z[2]$ to the imaginary part. The components $z[1]$ and $z[2]$ must be of type `t_INT`, `t_REAL` or `t_FRAC`. For historical reasons `t_INTMOD` and `t_PADIC` are also allowed (the latter for $p = 2$ or congruent to 3 mod 4 only), but one should rather use the more general `t_POLMOD` construction.

4.5.7 Type `t_PADIC` (p -adic numbers). this type has a second codeword $z[1]$ which contains the following information: the p -adic precision (the exponent of p modulo which the p -adic unit corresponding to z is defined if z is not 0), i.e. one less than the number of significant p -adic digits, and the exponent of z . This information can be handled using the following functions:

`long precp(GEN z)` returns the p -adic precision of z . This is 0 if $z = 0$.

`void setprecp(GEN z, long l)` sets the p -adic precision of z to l .

`long valp(GEN z)` returns the p -adic valuation of z (i.e. the exponent). This is defined even if z is equal to 0.

`void setvalp(GEN z, long e)` sets the p -adic valuation of z to e .

In addition to this codeword, $z[2]$ points to the prime p , $z[3]$ points to $p^{\text{precp}(z)}$, and $z[4]$ points to `at_INT` representing the p -adic unit attached to z modulo $z[3]$ (and to zero if z is zero). To summarize, if $z \neq 0$, we have the equality:

$$z = p^{\text{valp}(z)} * (z[4] + O(z[3])), \quad \text{where} \quad z[3] = O(p^{\text{precp}(z)}).$$

4.5.8 Type `t_QUAD` (quadratic number). `z[1]` points to the canonical polynomial P defining the quadratic field (as output by `quadpoly`), `z[2]` to the “real part” and `z[3]` to the “imaginary part”. The latter are of type `t_INT`, `t_FRAC`, `t_INTMOD`, or `t_PADIC` and are to be taken as the coefficients of z with respect to the canonical basis $(1, X)$ of $\mathbf{Q}[X]/(P(X))$. Exact complex numbers may be implemented as quadratics, but `t_COMPLEX` is in general more versatile (`t_REAL` components are allowed) and more efficient.

Operations involving a `t_QUAD` and `t_COMPLEX` are implemented by converting the `t_QUAD` to a `t_REAL` (or `t_COMPLEX` with `t_REAL` components) to the accuracy of the `t_COMPLEX`. As a consequence, operations between `t_QUAD` and *exact* `t_COMPLEX`s are not allowed.

4.5.9 Type `t_POLMOD` (polmod). as for `t_INTMOD`s, `z[1]` points to the modulus, and `z[2]` to a polynomial representing the class of z . Both must be of type `t_POL` in the same variable, satisfying the inequality $\deg z[2] < \deg z[1]$. However, `z[2]` is allowed to be a simplification of such a polynomial, e.g. a scalar. This is tricky considering the hierarchical structure of the variables; in particular, a polynomial in variable of *lesser* priority (see Section 4.6) than the modulus variable is valid, since it is considered as the constant term of a polynomial of degree 0 in the correct variable. On the other hand a variable of *greater* priority is not acceptable.

4.5.10 Type `t_POL` (polynomial). this type has a second codeword. It contains a “*sign*”: 0 if the polynomial is equal to 0, and 1 if not (see however the important remark below) and a *variable number* (e.g. 0 for x , 1 for y , etc...).

These data can be handled with the following macros: `signe` and `setsigne` as for `t_INT` and `t_REAL`,

`long varn(GEN z)` returns the variable number of the object z ,

`void setvarn(GEN z, long v)` sets the variable number of z to v .

The variable numbers encode the relative priorities of variables, we will give more details in Section 4.6. Note also the function `long gvar(GEN z)` which tries to return a variable number for z , even if z is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE`, which has lower priority than any other variable number.

The components `z[2]`, `z[3]`,...`z[lg(z)-1]` point to the coefficients of the polynomial *in ascending order*, with `z[2]` being the constant term and so on.

For a `t_POL` of non-zero sign, `degpol`, `leading_coeff`, `constant_coeff`, return its degree, and a pointer to the leading, resp. constant, coefficient with respect to the main variable. Note that no copy is made on the PARI stack so the returned value is not safe for a basic `gerepile` call. Applied to any other type than `t_POL`, the result is unspecified. Those three functions are still defined when the sign is 0, see Section 5.2.7 and Section 10.6.

`long degree(GEN x)` returns the degree of x with respect to its main variable even when x is not a polynomial (a rational function for instance). By convention, the degree of a zero polynomial is -1 .

Important remark. The leading coefficient of a `t_POL` may be equal to zero:

- it is not allowed to be an exact rational 0, such as `gen_0`;
- an exact non-rational 0, like `Mod(0,2)`, is possible for constant polynomials, i.e. of length 3 and no other coefficient: this carries information about the base ring for the polynomial;
- an inexact 0, like `0.E-38` or `0(3^5)`, is always possible. Inexact zeroes do not correspond to an actual 0, but to a very small coefficient according to some metric; we keep them to give information on how much cancellation occurred in previous computations.

A polynomial disobeying any of these rules is an invalid *unnormalized* object. We advise *not* to use low-level constructions to build a `t_POL` coefficient by coefficient, such as

```
GEN T = cgetg(4, t_POL);
T[1] = evalvarn(0);
gel(T, 2) = x;
gel(T, 3) = y;
```

But if you do and it is not clear whether the result will be normalized, call

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. This function sets `signe` (to 0 or 1) properly.

Caveat. A consequence of the remark above is that zero polynomials are characterized by the fact that their sign is 0. It is in general incorrect to check whether `lg(x)` is 2 or `degpol(x) < 0`, although both tests are valid when the coefficient types are under control: for instance, when they are guaranteed to be `t_INTs` or `t_FRACs`. The same remark applies to `t_SERs`.

4.5.11 Type `t_SER` (power series). This type also has a second codeword, which encodes a “*sign*”, i.e. 0 if the power series is 0, and 1 if not, a *variable number* as for polynomials, and an *exponent*. This information can be handled with the following functions: **signe**, **setsigne**, **varn**, **setvarn** as for polynomials, and **valp**, **setvalp** for the exponent as for p -adic numbers. Beware: do *not* use **expo** and **setexpo** on power series.

The coefficients `z[2]`, `z[3]`, ... `z[lg(z)-1]` point to the coefficients of `z` in ascending order. As for polynomials (see remark there), the sign of a `t_SER` is 0 if and only all its coefficients are equal to 0. (The leading coefficient cannot be an integer 0.) A series whose coefficients are integers equal to zero is represented as $O(x^n)$ (**zéroser**(`vx`, `n`)). A series whose coefficients are exact zeroes, but not all of them integers (e.g. an `t_INTMOD` such as `Mod(0,2)`) is represented as $z * x^{n-1} + O(x^n)$, where `z` is the 0 of the base ring, as per **Rg_get_0**.

Note that the exponent of a power series can be negative, i.e. we are then dealing with a Laurent series (with a finite number of negative terms).

4.5.12 Type `t_RFRAC` (rational function). `z[1]` points to the numerator n , and `z[2]` on the denominator d . The denominator must be of type `t_POL`, with variable of higher priority than the numerator. The numerator n is not an exact 0 and $(n, d) = 1$ (see **gred_rfac2**).

4.5.13 Type `t_QFR` (indefinite binary quadratic form). `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form and are of type `t_INT`. `z[4]` is Shanks’s distance function, and must be of type `t_REAL`.

4.5.14 Type `t_QFI` (definite binary quadratic form). `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form. All three are of type `t_INT`.

4.5.15 Type `t_VEC` and `t_COL` (vector). `z[1]`, `z[2]`, ..., `z[lg(z)-1]` point to the components of the vector.

4.5.16 Type `t_MAT` (matrix). `z[1]`, `z[2]`, ..., `z[lg(z)-1]` point to the column vectors of `z`, i.e. they must be of type `t_COL` and of the same length.

4.5.17 Type `t_VECSMALL` (vector of small integers). `z[1]`, `z[2]`, ..., `z[lg(z)-1]` are ordinary signed long integers. This type is used instead of a `t_VEC` of `t_INTs` for efficiency reasons, for instance to implement efficiently permutations, polynomial arithmetic and linear algebra over small finite fields, etc.

4.5.18 Type `t_STR` (character string).

`char * GSTR(z) (= (z+1))` points to the first character of the (NULL-terminated) string.

4.5.19 Type `t_ERROR` (error context). This type holds error messages, as well as details about the error, as returned by the exception handling system. The second codeword `z[1]` contains the error type (an `int`, as passed to `pari_err`). The subsequent words `z[2]`, ..., `z[lg(z)-1]` are GENs containing additional data, depending on the error type.

4.5.20 Type `t_CLOSURE` (closure). This type holds GP functions and closures, in compiled form. The internal detail of this type is subject to change each time the GP language evolves. Hence we do not describe it here and refer to the Developer's Guide. However functions to create or to evaluate `t_CLOSUREs` are documented in Section 12.1.

`long closure_arity(GEN C)` returns the arity of the `t_CLOSURE`.

`long closure_is_variadic(GEN C)` returns 1 if the closure `C` is variadic, 0 else.

4.5.21 Type `t_INFINITY` (infinity).

This type has a single `t_INT` component, which is either 1 or -1 , corresponding to $+\infty$ and $-\infty$ respectively.

`GEN mkmoo()` returns $-\infty$

`GEN mkoo()` returns ∞

`long inf_get_sign(GEN x)` returns 1 if x is $+\infty$, and -1 if x is $-\infty$.

4.5.22 Type `t_LIST` (list). this type was introduced for specific `gp` use and is rather inefficient compared to a straightforward linked list implementation (it requires more memory, as well as many unnecessary copies). Hence we do not describe it here and refer to the Developer's Guide.

Implementation note. For the types including an exponent (or a valuation), we actually store a biased non-negative exponent (bit-ORing the biased exponent to the codeword), obtained by adding a constant to the true exponent: either `HIGHXPOBIT` (for `t_REAL`) or `HIGHVALPBIT` (for `t_PADIC` and `t_SER`). Of course, this is encapsulated by the exponent/valuation-handling macros and needs not concern the library user.

4.6 PARI variables.

4.6.1 Multivariate objects.

We now consider variables and formal computations. As we have seen in Section 4.5, the codewords for types `t_POL` and `t_SER` encode a “variable number”. This is an integer, ranging from 0 to `MAXVARN`. Relative priorities may be ascertained using

```
int varncmp(long v, long w)
```

which is > 0 , $= 0$, < 0 whenever v has lower, resp. same, resp. higher priority than w .

The way an object is considered in formal computations depends entirely on its “principal variable number” which is given by the function

```
long gvar(GEN z)
```

which returns a variable number for z , even if z is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE` which has lower priority than any valid variable number. The variable number of a recursive type which is not a polynomial or power series is the variable number with highest priority among its components. But for polynomials and power series only the “outermost” number counts (we directly access `varn(x)` in the codewords): the representation is not symmetrical at all.

Under `gp`, one needs not worry too much since the interpreter defines the variables as it sees them* and do the right thing with the polynomials produced.

But in library mode, they are tricky objects if you intend to build polynomials yourself (and not just let PARI functions produce them, which is less efficient). For instance, it does not make sense to have a variable number occur in the components of a polynomial whose main variable has a lower priority, even though PARI cannot prevent you from doing it.

4.6.2 Creating variables. A basic difficulty is to “create” a variable. Some initializations are needed before you can use a given integer v as a variable number.

Initially, this is done for 0 and 1 (the variables `x` and `y` under `gp`), and 2, ..., 9 (printed as `t2`, ... `t9`), with decreasing priority.

4.6.2.1 User variables. When the program starts, `x` (number 0) and `y` (number 1) are the only available variables, numbers 2 to 9 (decreasing priority) are reserved for building polynomials with predictable priorities.

To define further ones, you may use

```
GEN varhigher(const char *s)
```

```
GEN varlower(const char *s)
```

to recover a monomial of degree 1 in a new variable, which is guaranteed to have higher (resp. lower) priority than all existing ones at the time of the function call. The variable is printed as s , but is not part of GP’s interpreter: it is not a symbol bound to a value.

* The first time a given identifier is read by the GP parser a new variable is created, and it is assigned a strictly lower priority than any variable in use at this point. On startup, before any user input has taken place, ‘`x`’ is defined in this way and has initially maximal priority (and variable number 0).

On the other hand

`long fetch_user_var(char *s)`: inspects the user variable whose name is the string pointed to by `s`, creating it if needed, and returns its variable number.

```
long v = fetch_user_var("y");
GEN gy = pol_x(v);
```

The function raises an exception if the name is already in use for an `installed` or built-in function, or an alias. This function is mostly useless since it returns a variable with unpredictable priority. Don't use it to create new variables.

Caveat. You can use `gp_read_str` (see Section 4.7.1) to execute a GP command and create GP variables on the fly as needed:

```
GEN gy = gp_read_str("'y"); /* returns pol_x(v), for some v */
long v = varn(gy);
```

But please note the quote `'y` in the above. Using `gp_read_str("y")` might work, but is dangerous, especially when programming functions to be used under `gp`. The latter reads the value of `y`, as *currently* known by the `gp` interpreter, possibly creating it in the process. But if `y` has been modified by previous `gp` commands (e.g. `y = 1`), then the value of `gy` is not what you expected it to be and corresponds instead to the current value of the `gp` variable (e.g. `gen_1`).

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered `v`. Returns `NULL` if that variable number is unknown to the interpreter, e.g. it is a user variable. Note that this may not be the same as `pol_x(v)` if assignments have been performed in the interpreter.

4.6.2.2 Temporary variables. You can create temporary variables using

`long fetch_var()` returns a new variable with *lower* priority than any variable currently in use.

`long fetch_var_higher()` returns a new variable with *higher* priority than any variable currently in use.

After the statement `v = fetch_var()`, you can use `pol_1(v)` and `pol_x(v)`. The variables created in this way have no identifier assigned to them though, and are printed as `tnumber`. You can assign a name to a temporary variable, after creating it, by calling the function

```
void name_var(long n, char *s)
```

after which the output machinery will use the name `s` to represent the variable number `n`. The GP parser will *not* recognize it by that name, however, and calling this on a variable known to `gp` raises an error. Temporary variables are meant to be used as free variables to build polynomials and power series, and you should never assign values or functions to them as you would do with variables under `gp`. For that, you need a user variable.

All objects created by `fetch_var` are on the heap and not on the stack, thus they are not subject to standard garbage collecting (they are not destroyed by a `gerepile` or `avma = ltop` statement). When you do not need a variable number anymore, you can delete it using

```
long delete_var()
```

which deletes the *latest* temporary variable created and returns the variable number of the previous one (or simply returns 0 if none remain). Of course you should make sure that the deleted variable does not appear anywhere in the objects you use later on. Here is an example:

```

long first = fetch_var();
long n1 = fetch_var();
long n2 = fetch_var(); /* prepare three variables for internal use */
...
/* delete all variables before leaving */
do { num = delete_var(); } while (num && num <= first);

```

The (dangerous) statement

```
while (delete_var()) /* empty */;
```

removes all temporary variables in use.

4.6.3 Comparing variables.

Let us go back to `varncmp`. There is an interesting corner case, when one of the compared variables (from `gvar`, say) is `NO_VARIABLE`. In this case, `varncmp` declares it has lower priority than any other variable; of course, comparing `NO_VARIABLE` with itself yields 0 (same priority);

In addition to `varncmp` we have

`long varnmax(long v, long w)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the highest priority. This function always returns a valid variable number unless it is comparing `NO_VARIABLE` to itself.

`long varnmin(long x, long y)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the lowest priority. Note that when comparing a true variable with `NO_VARIABLE`, this function returns `NO_VARIABLE`, which is not a valid variable number.

4.7 Input and output.

Two important aspects have not yet been explained which are specific to library mode: input and output of PARI objects.

4.7.1 Input.

For input, PARI provides several powerful high level functions which enable you to input your objects as if you were under `gp`. In fact, it *is* essentially the GP syntactical parser.

There are two similar functions available to parse a string:

```
GEN gp_read_str(const char *s)
```

```
GEN gp_read_str_multiline(const char *s, char *last)
```

Both functions read the whole string `s`. The function `gp_read_str` ignores newlines: it assumes that the input is one expression and returns the result of this expression.

The function `gp_read_str_multiline` processes the text in the same way as the GP command `read`: newlines are significant and can be used to separate expressions. The return value is that of the last non-empty expression evaluated.

In `gp_read_str_multiline`, if `last` is non-NULL, then `*last` receives the last character from the *filtered* input: this can be used to check if the last character was a semi-colon (to hide the output in interactive usage). If (and only if) the input contains no statements, then `*last` is set to 0.

For both functions, `gp`'s metacommands *are* recognized.

Note. The obsolete form

```
GEN readseq(char *t)
```

still exists for backward compatibility (assumes filtered input, without spaces or comments). Don't use it.

To read a GEN from a file, you can use the simpler interface

```
GEN gp_read_stream(FILE *file)
```

which reads a character string of arbitrary length from the stream `file` (up to the first complete expression sequence), applies `gp_read_str` to it, and returns the resulting GEN. This way, you do not have to worry about allocating buffers to hold the string. To interactively input an expression, use `gp_read_stream(stdin)`.

Finally, you can read in a whole file, as in GP's `read` statement

```
GEN gp_read_file(char *name)
```

As usual, the return value is that of the last non-empty expression evaluated. There is one technical exception: if `name` is a *binary* file (from `writebin`) containing more than one object, a `t_VEC` containing them all is returned. This is because binary objects bypass the parser, hence reading them has no useful side effect.

4.7.2 Output to screen or file, output to string.

General output functions return nothing but print a character string as a side effect. Low level routines are available to write on PARI output stream `pari_outfile` (`stdout` by default):

`void pari_putc(char c):` write character `c` to the output stream.

`void pari_puts(char *s):` write `s` to the output stream.

`void pari_flush():` flush output stream; most streams are buffered by default, this command makes sure that all characters output so are actually written.

`void pari_printf(const char *fmt, ...):` the most versatile such function. `fmt` is a character string similar to the one `printf` uses. In there, `%` characters have a special meaning, and describe how to print the remaining operands. In addition to the standard format types (see the GP function `printf`), you can use the *length modifier* `P` (for PARI of course!) to specify that an argument is a GEN. For instance, the following are valid conversions for a GEN argument

```
%Ps      convert to char* (will print an arbitrary GEN)
%P.10s   convert to char*, truncated to 10 chars
%P.2f    convert to floating point format with 2 decimals
%P4d     convert to integer, field width at least 4
```

```
pari_printf("x[%d] = %Ps is not invertible!\n", i, gel(x,i));
```

Here `i` is an `int`, `x` a GEN which is not a leaf (presumably a vector, or a polynomial) and this would insert the value of its *i*-th GEN component: `gel(x,i)`.

Simple but useful variants to `pari_printf` are

`void output(GEN x)` prints `x` in raw format, followed by a newline and a buffer flush. This is more or less equivalent to

```
pari_printf("%Ps\n", x);
```

```
pari_flush();
```

`void outmat(GEN x)` as above except if x is a `t_MAT`, in which case a multi-line display is used to display the matrix. This is prettier for small dimensions, but quickly becomes unreadable and cannot be pasted and reused for input. If all entries of x are small integers, you may use the recursive features of `%Pd` and obtain the same (or better) effect with

```
pari_printf("%Pd\n", x);
pari_flush();
```

A variant like `%5Pd` would improve alignment by imposing 5 chars for each coefficient. Similarly if all entries are to be converted to floats, a format like `%5.1Pf` could be useful.

These functions write on (PARI's idea of) standard output, and must be used if you want your functions to interact nicely with `gp`. In most programs, this is not a concern and it is more flexible to write to an explicit `FILE*`, or to recover a character string:

`void pari_fprintf(FILE *file, const char *fmt, ...)` writes the remaining arguments to stream `file` according to the format specification `fmt`.

`char* pari_sprintf(const char *fmt, ...)` produces a string from the remaining arguments, according to the PARI format `fmt` (see `printf`). This is the `libpari` equivalent of `Strprintf`, and returns a `malloc`'ed string, which must be freed by the caller. Note that contrary to the analogous `sprintf` in the `libc` you do not provide a buffer (leading to all kinds of buffer overflow concerns); the function provided is actually closer to the GNU extension `asprintf`, although the latter has a different interface.

Simple variants of `pari_sprintf` convert a `GEN` to a `malloc`'ed ASCII string, which you must still `free` after use:

`char* GENtostr(GEN x)`, using the current default output format (`prettymat` by default).

`char* GENtoTeXstr(GEN x)`, suitable for inclusion in a `TeX` file.

Note that we have `va_list` analogs of the functions of `printf` type seen so far:

```
void pari_vprintf(const char *fmt, va_list ap)
```

```
void pari_vfprintf(FILE *file, const char *fmt, va_list ap)
```

```
char* pari_vsprintf(const char *fmt, va_list ap)
```

4.7.3 Errors.

If you want your functions to issue error messages, you can use the general error handling routine `pari_err`. The basic syntax is

```
pari_err(e_MISC, "error message");
```

This prints the corresponding error message and exit the program (in library mode; go back to the `gp` prompt otherwise). You can also use it in the more versatile guise

```
pari_err(e_MISC, format, ...);
```

where `format` describes the format to use to write the remaining operands, as in the `pari_printf` function. For instance:

```
pari_err(e_MISC, "x[%d] = %Ps is not invertible!", i, gel(x,i));
```

The simple syntax seen above is just a special case with a constant format and no remaining arguments. The general syntax is

```
void pari_err(numerr, ...)
```

where `numerr` is a codeword which specifies the error class and what to do with the remaining arguments and what message to print. For instance, if `x` is a `GEN` with internal type `t_STR`, say, `pari_err(e_TYPE, "extgcd", x)` prints the message:

```
*** incorrect type in extgcd (t_STR),
```

See Section 11.4 for details. In the libpari code itself, the general-purpose `e_MISC` is used sparingly: it is so flexible that the corresponding error contexts (`t_ERROR`) become hard to use reliably. Other more rigid error types are generally more useful: for instance the error context attached to the `e_TYPE` exception above is precisely documented and contains `"extgcd"` and `x` (not only its type) as readily available components.

4.7.4 Warnings.

To issue a warning, use

```
void pari_warn(warnerr, ...) In that case, of course, we do not abort the computation, just print the requested message and go on. The basic example is
```

```
pari_warn(warner, "Strategy 1 failed. Trying strategy 2")
```

which is the exact equivalent of `pari_err(e_MISC, ...)` except that you certainly do not want to stop the program at this point, just inform the user that something important has occurred; in particular, this output would be suitably highlighted under `gp`, whereas a simple `printf` would not.

The valid *warning* keywords are `warner` (general), `warnprec` (increasing precision), `warnmem` (garbage collecting) and `warnfile` (error in file operation), used as follows:

```
pari_warn(warnprec, "bnfinit", newprec);
pari_warn(warnmem, "bnfinit");
pari_warn(warnfile, "close", "afile"); /* error when closing "afile" */
```

4.7.5 Debugging output.

For debugging output, you can use the standard output functions, `output` and `pari_printf` mainly. Corresponding to the `gp` metacommand `\x`, you can also output the hexadecimal tree attached to an object:

```
void dbgGEN(GEN x, long nb = -1), displays the recursive structure of x. If nb = -1, the full structure is printed, otherwise the leaves (non-recursive components) are truncated to nb words.
```

The function `output` is vital under debuggers, since none of them knows how to print PARI objects by default. Seasoned PARI developers add the following `gdb` macro to their `.gdbinit`:

```
define i
  call output((GEN)$arg0)
end
```

Typing `i x` at a breakpoint in `gdb` then prints the value of the `GEN x` (provided the optimizer has not put it into a register, but it is rarely a good idea to debug optimized code).

The global variables **DEBUGLEVEL** and **DEBUGMEM** (corresponding to the default **debug** and **debugmem**) are used throughout the PARI code to govern the amount of diagnostic and debugging output, depending on their values. You can use them to debug your own functions, especially if you `install` the latter under `gp`.

`void dbg_pari_heap(void)` print debugging statements about the PARI stack, heap, and number of variables used. Corresponds to `\s` under `gp`.

4.7.6 Timers and timing output.

To handle timings in a reentrant way, PARI defines a dedicated data type, `pari_timer`, together with the following methods:

`void timer_start(pari_timer *T)` start (or reset) a timer.

`long timer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Resets the timer as a side effect.

`long timer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer.

`long timer_printf(pari_timer *T, char *format, ...)` This diagnostics function is equivalent to the following code

```
err_printf("Time ")
... prints remaining arguments according to format ...
err_printf(": %ld", timer_delay(T));
```

Resets the timer as a side effect.

They are used as follows:

```
pari_timer T;
timer_start(&T); /* initialize timer */
...
printf("Total time: %ldms\n", timer_delay(&T));
```

or

```
pari_timer T;
timer_start(&T);
for (i = 1; i < 10; i++) {
...
timer_printf(&T, "for i = %ld (L[i] = %Ps)", i, gel(L,i));
}
```

The following functions provided the same functionality, in a non-reentrant way, and are now deprecated.

`long timer(void)`

`long timer2(void)`

`void msgtimer(const char *format, ...)`

The following function implements `gp`'s timer and should not be used in `libpari` programs: `long gettime(void)` equivalent to `timer_delay(T)` attached to a private timer `T`.

4.8 Iterators, Numerical integration, Sums, Products.

4.8.1 Iterators. Since it is easier to program directly simple loops in library mode, some GP iterators are mainly useful for GP programming. Here are the others:

- `fordiv` is a trivial iteration over a list produced by `divisors`.
- `forell` and `forsubgroup` are currently not implemented as an iterator but as a procedure with callbacks.

`void forell(void *E, long fun(void*, GEN), GEN a, GEN b)` goes through the same curves as `forell(ell,a,b,)`, calling `fun(E, ell)` for each curve `ell`, stopping if `fun` returns a non-zero value.

`void forsubgroup(void *E, long fun(void*, GEN), GEN G, GEN B)` goes through the same subgroups as `forsubgroup(H = G, B,)`, calling `fun(E, H)` for each subgroup `H`, stopping if `fun` returns a non-zero value.

- `forprime`, for which we refer you to the next subsection.
- `forcomposite`, we provide an iterator over composite integers:

`int forcomposite(forcomposite_t *T, GEN a, GEN b)` initialize an iterator T over composite integers in $[a, b]$; over composites $\geq a$ if $b = \text{NULL}$. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise.

`GEN forcomposite_next(forcomposite_t *T)` returns the next composite in the range, assuming that T was initialized by `forcomposite_init`.

- `forvec`, for which we provide a convenient iterator. To initialize the analog of `forvec(X = v, ..., flag)`, call

`int forvec_init(forvec_t *T, GEN v, long flag)` initialize an iterator T over the vectors generated by `forvec(X = v, ..., flag)`. This returns 0 if this vector list is empty, and 1 otherwise.

`GEN forvec_next(forvec_t *T)` returns the next element in the `forvec` sequence, or `NULL` if we are done. The return value must be used immediately or copied since the next call to the iterator destroys it: the relevant vector is updated in place. The iterator works hard to not use up PARI stack, and is more efficient when all lower bounds in the initialization vector v are integers. In that case, the cost is linear in the number of tuples enumerated, and you can expect to run over more than 10^9 tuples per minute. If speed is critical and all integers involved would fit in C longs, write a simple direct backtracking algorithm yourself.

- `forpart` is a variant of `forvec` which iterates over partitions. See the documentation of the `forpart` GP function for details. This function is available as a loop with callbacks:

`void forpart(void *data, long (*call)(void*, GEN), long k, GEN a, GEN n)`

It is also available as an iterator:

`void forpart_init(forpart_t *T, long k, GEN a, GEN n)` initializes an iterator over the partitions of k , with length restricted by n , and components restricted by a , either of which can be set to `NULL` to run without restriction.

`GEN forpart_next(forpart_t *T)` returns the next partition, or `NULL` when all partitions have been exhausted.

GEN `forpart_prev(forpart_t *T)` returns the previous partition, or NULL when all partitions have been exhausted.

In both cases, the partition must be used or copied before the next call since it is returned from a state array which will be modified in place. You may *not* mix calls to `forpart_next` and `forpart_prev`: the first one called determines the ordering used to iterate over the partitions; you can not go back since the `forpart_t` structure is used in incompatible ways.

- `forperm` to loop over permutations of k . See the documentation of the `forperm` GP function for details. This function is available as an iterator:

`void forperm_init(forperm_t *T, GEN k)` initializes an iterator over the permutations of k (`t_INT`, `t_VEC` or `t_VECSMALL`).

GEN `forperm_next(forperm_t *T)` returns the next permutation as a `t_VECSMALL` or NULL when all permutations have been exhausted. The permutation must be used or copied before the next call since it is returned from a state array which will be modified in place.

- `forsubset` to loop over subsets. See the documentation of the `forsubset` GP function for details. This function is available as two iterators:

`void forallsubset_init(forsubset_t *T, long n)`

`void forksubset_init(forsubset_t *T, long n, long k)`

It is also available in generic form:

`void forsubset_init(forsubset_t *T, GEN nk)` where `nk` is either a `t_INT` n or a `t_VEC` with two integral components $[n, k]$.

In all three cases, GEN `forsubset_next(forsubset_t *T)` returns the next subset as a `t_VECSMALL` or NULL when all subsets have been exhausted.

4.8.2 Iterating over primes.

The library provides a high-level iterator, which stores its (private) data in a `struct forprime_t` and runs over arbitrary ranges of primes, without ever overflowing.

The iterator has two flavors, one providing the successive primes as `ulongs`, the other as GEN. They are initialized as follows, where we expect to run over primes $\geq a$ and $\leq b$:

`int forprime_init(forprime_t *T, GEN a, GEN b)` for the GEN variant, where $b = \text{NULL}$ means $+\infty$.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)` for the `ulong` variant, where $b = \text{ULONG_MAX}$ means we will run through all primes representable in a `ulong` type.

Both variant return 1 on success, and 0 if the iterator would run over an empty interval (if $a > b$, for instance). They allocate the `forprime_t` data structure on the PARI stack.

The successive primes are then obtained using

GEN `forprime_next(forprime_t *T)`, returns NULL if no more primes are available in the interval.

`ulong u_forprime_next(forprime_t *T)`, returns 0 if no more primes are available in the interval.

These two functions leave alone the PARI stack, and write their state information in the preallocated `forprime_t` struct. The typical usage is thus:

```

forprime_t T;
GEN p;
pari_sp av = avma, av2;

forprime_init(&T, gen_2, stoi(1000));
av2 = avma;
while ( (p = forprime_next(&T)) )
{
    ...
    if ( prime_is_OK(p) ) break;
    avma = av2; /* delete garbage accumulated in this iteration */
}
avma = av; /* delete all */

```

Of course, the final `avma = av` could be replaced by a `gerepile` call. Beware that swapping the `av2 = avma` and `forprime_init` call would be incorrect: the first `avma = av2` would delete the `forprime_t` structure!

4.8.3 Numerical analysis.

Numerical routines code a function (to be integrated, summed, zeroed, etc.) with two parameters named

```

void *E;
GEN (*eval)(void*, GEN)

```

The second is meant to contain all auxiliary data needed by your function. The first is such that `eval(x, E)` returns your function evaluated at `x`. For instance, one may code the family of functions $f_t : x \rightarrow (x + t)^2$ via

```

GEN fun(void *t, GEN x) { return gsqr(gadd(x, (GEN)t)); }

```

One can then integrate f_1 between a and b with the call

```

intnum((void*)stoi(1), &fun, a, b, NULL, prec);

```

Since you can set `E` to a pointer to any `struct` (typecast to `void*`) the above mechanism handles arbitrary functions. For simple functions without extra parameters, you may set `E = NULL` and ignore that argument in your function definition.

4.9 Catching exceptions.

4.9.1 Basic use.

PARI provides a mechanism to trap exceptions generated via `pari_err` using the `pari_CATCH` construction. The basic usage is as follows

```
pari_CATCH(err_code) {
    recovery branch
}
pari_TRY {
    main branch
}
pari_ENDCATCH
```

This fragment executes the main branch, then the recovery branch *if* exception `err_code` is thrown, e.g. `e_TYPE`. See Section 11.4 for the description of all error classes. The special error code `CATCH_ALL` is available to catch all errors.

One can replace the `pari_TRY` keyword by `pari_RETRY`, in which case once the recovery branch is run, we run the main branch again, still catching the same exceptions.

Restrictions.

- Such constructs can be nested without adverse effect, the innermost handler catching the exception.
- It is *valid* to leave either branch using `pari_err`.
- It is *invalid* to use C flow control instructions (`break`, `continue`, `return`) to directly leave either branch without seeing the `pari_ENDCATCH` keyword. This would leave an invalid structure in the exception handler stack, and the next exception would crash.
- In order to leave using `break`, `continue` or `return`, one must precede the keyword by a call to

`void pari_CATCH_reset()` disable the current handler, allowing to leave without adverse effect.

4.9.2 Advanced use.

In the recovery branch, the exception context can be examined via the following helper routines:

`GEN pari_err_last()` returns the exception context, as a `t_ERROR`. The exception `E` returned by `pari_err_last` can be rethrown, using

```
pari_err(0, E);
```

`long err_get_num(GEN E)` returns the error symbolic name. E.g `e_TYPE`.

`GEN err_get_compo(GEN E, long i)` error *i*-th component, as documented in Section 11.4.

For instance

```
pari_CATCH(CATCH_ALL) { /* catch everything */
    GEN x, E = pari_err_last();
    long code = err_get_num(E);
    if (code != e_INV) pari_err(0, E); /* unexpected error, rethrow */
}
```

```

x = err_get_compo(E, 2);
/* e_INV has two components, 1: function name 2: non-invertible x */
if (typ(x) != t_INTMOD) pari_err(0, E); /* unexpected type, rethrow */
pari_CATCH_reset();
return x; /* leave ! */
...
} pari_TRY {
    main branch
}
pari_ENDCATCH

```

4.10 A complete program.

Now that the preliminaries are out of the way, the best way to learn how to use the library mode is to study a detailed example. We want to write a program which computes the gcd of two integers, together with the Bezout coefficients. We shall use the standard quadratic algorithm which is not optimal but is not too far from the one used in the PARI function **bezout**.

Let x, y two integers and initially $\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To apply the ordinary Euclidean algorithm to the right hand side, multiply the system from the left by $\begin{pmatrix} 0 & 1 \\ 1 & -q \end{pmatrix}$, with $q = \text{floor}(x/y)$. Iterate until $y = 0$ in the right hand side, then the first line of the system reads

$$s_x x + s_y y = \text{gcd}(x, y).$$

In practice, there is no need to update s_y and t_y since $\text{gcd}(x, y)$ and s_x are enough to recover s_y . The following program is now straightforward. A couple of new functions appear in there, whose description can be found in the technical reference manual in Chapter 5, but whose meaning should be clear from their name and the context.

This program can be found in `examples/extgcd.c` together with a proper Makefile. You may ignore the first comment

```

/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

```

which instruments the program so that `gp2c-run extgcd.c` can import the `extgcd()` routine into an instance of the `gp` interpreter (under the name `gcdex`). See the `gp2c` manual for details.

```

#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
  pari_sp av = avma;
  GEN ux = gen_1, vx = gen_0, a = A, b = B;
  if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
  if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
  if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
  while (!gequal0(b))
  {
    GEN r, q = dvmdii(a, b, &r), v = vx;
    vx = subii(ux, mulii(q, vx));
    ux = v; a = b; b = r;
  }
  *U = ux;
  *V = diviexact( subii(a, mulii(A,ux)), B );
  gerepileall(av, 3, &a, U, V); return a;
}

int
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}

```

For simplicity, the inner loop does not include any garbage collection, hence memory use is quadratic in the size of the inputs instead of linear. Here is a better version of that loop:

```

pari_sp av = avma;
...
while (!gequal0(b))
{
  GEN r, q = dvmdii(a, b, &r), v = vx;
  vx = subii(ux, mulii(q, vx));
  ux = v; a = b; b = r;
  if (gc_needed(av,1))
    gerepileall(av, 4, &a, &b, &ux, &vx);
}

```

}

Chapter 5: Technical Reference Guide: the basics

In the following chapters, we describe all public low-level functions of the PARI library. These include specialized functions for handling all the PARI types. Simple higher level functions, such as arithmetic or transcendental functions, are described in Chapter 3 of the GP user's manual; we will eventually see more general or flexible versions in the chapters to come. A general introduction to the major concepts of PARI programming can be found in Chapter 4, which you should really read first.

We shall now study specialized functions, more efficient than the library wrappers, but sloppier on argument checking and damage control; besides speed, their main advantage is to give finer control about the inner workings of generic routines, offering more options to the programmer.

Important advice. Generic routines eventually call lower level functions. Optimize your algorithms first, not overhead and conversion costs between PARI routines. For generic operations, use generic routines first; do not waste time looking for the most specialized one available unless you identify a genuine bottleneck, or you need some special behavior the generic routine does not offer. The PARI source code is part of the documentation; look for inspiration there.

The type `long` denotes a `BITS_IN_LONG`-bit signed long integer (32 or 64 bits). The type `ulong` is defined as `unsigned long`. The word *stack* always refer to the PARI stack, allocated through an initial `pari_init` call. Refer to Chapters 1–2 and 4 for general background.

We shall often refer to the notion of *shallow* function, which means that some components of the result may point to components of the input, which is more efficient than a *deep* copy (full recursive copy of the object tree). Such outputs are not suitable for `gerepileupto` and particular care must be taken when garbage collecting objects which have been input to shallow functions: corresponding outputs also become invalid and should no longer be accessed.

A function is *not stack clean* if it leaves intermediate data on the stack besides its output, for efficiency reasons.

5.1 Initializing the library.

The following functions enable you to start using the PARI functions in a program, and cleanup without exiting the whole program.

5.1.1 General purpose.

`void pari_init(size_t size, ulong maxprime)` initialize the library, with a stack of `size` bytes and a prime table up to the maximum of `maxprime` and 2^{16} . Unless otherwise mentioned, no PARI function will function properly before such an initialization.

`void pari_close(void)` stop using the library (assuming it was initialized with `pari_init`) and frees all allocated objects.

5.1.2 Technical functions.

`void pari_init_opts(size_t size, ulong maxprime, ulong opts)` as `pari_init`, more flexible. `opts` is a mask of flags among the following:

`INIT_JMPm`: install PARI error handler. When an exception is raised, the program is terminated with `exit(1)`.

`INIT_SIGm`: install PARI signal handler.

`INIT_DFTm`: initialize the `GP_DATA` environment structure. This one *must* be enabled once. If you close `pari`, then restart it, you need not reinitialize `GP_DATA`; if you do not, then old values are restored.

`INIT_noPRIMEm`: do not compute the prime table (ignore the `maxprime` argument). The user *must* call `pari_init_primes` later.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_init` to initialize the multi-thread engine. If this flag is set, `pari_mt_init()` will need to be called manually. See `examples/pari-mt.c` for an example.

`INIT_noINTGMPm`: do not install PARI-specific GMP memory functions. This option is ignored when the GMP library is not in use. You may install PARI-specific GMP memory functions later by calling

```
void pari_kernel_init(void)
```

and restore the previous values using

```
void pari_kernel_close(void)
```

This option should not be used without a thorough understanding of the problem you are trying to solve. The GMP memory functions are global variables used by the GMP library. If your program is linked with two libraries that require these variables to be set to different values, conflict ensues. To avoid a conflict, the proper solution is to record their values with `mp_get_memory_functions` and to call `mp_set_memory_functions` to restore the expected values each time the code switches from using one library to the other. Here is an example:

```
void>(*pari_alloc_ptr)(size_t);
void>(*pari_realloc_ptr)(void*, size_t, size_t);
void(*pari_free_ptr)(void*, size_t);
void(*otherlib_alloc_ptr)(size_t);
void(*otherlib_realloc_ptr)(void*, size_t, size_t);
void(*otherlib_free_ptr)(void*, size_t);

void init(void)
{
    pari_init(8000000, 500000);
    mp_get_memory_functions(&pari_alloc_ptr, &pari_realloc_ptr,
                          &pari_free_ptr);

    otherlib_init();
    mp_get_memory_functions(&otherlib_alloc_ptr, &otherlib_realloc_ptr,
                          &otherlib_free_ptr);
}

void function_that_use_pari(void)
{
```

```

    mp_set_memory_functions(pari_alloc_ptr,pari_realloc_ptr,
                           pari_free_ptr);
    /*use PARI functions*/
}
void function_that_use_otherlib(void)
{
    mp_set_memory_functions(otherlib_alloc_ptr,otherlib_realloc_ptr,
                           otherlib_free_ptr);
    /*use OTHERLIB functions*/
}

```

`void pari_close_opts(ulong init_opts)` as `pari_close`, for a library initialized with a mask of options using `pari_init_opts`. `opts` is a mask of flags among

`INIT_SIGm`: restore `SIG_DFL` default action for signals tampered with by PARI signal handler.

`INIT_DFTm`: frees the `GP_DATA` environment structure.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_close` to close the multi-thread engine. `INIT_noINTGMPm`: do not restore GMP memory functions.

`void pari_sig_init(void (*f)(int))` install the signal handler `f` (see `signal(2)`): the signals `SIGBUS`, `SIGFPE`, `SIGINT`, `SIGBREAK`, `SIGPIPE` and `SIGSEGV` are concerned.

`void pari_init_primes(ulong maxprime)` Initialize the PARI primes. This function is called by `pari_init(...,maxprime)`. It is provided for users calling `pari_init_opts` with the flag `INIT_noPRIMEm`.

`void pari_sighandler(int signum)` the actual signal handler that PARI uses. This can be used as argument to `pari_sig_init` or `signal(2)`.

`void pari_stackcheck_init(void *stackbase)` controls the system stack exhaustion checking code in the GP interpreter. This should be used when the system stack base address change or when the address seen by `pari_init` is too far from the base address. If `stackbase` is `NULL`, disable the check, else set the base address to `stackbase`. It is normally used this way

```

int thread_start (...)
{
    long first_item_on_the_stack;
    ...
    pari_stackcheck_init(&first_item_on_the_stack);
}

```

`int pari_daemon(void)` forks a PARI daemon, detaching from the main process group. The function returns 1 in the parent, and 0 in the forked son.

`void paristack_setsize(size_t rsize, size_t vsize)` sets the default `parisize` to `rsize` and the default `parisizemax` to `vsize`, and reallocate the stack to match these value, destroying its content. Generally used just after `pari_init`.

`void paristack_resize(ulong newsize)` changes the current stack size to `newsize` (double it if `newsize` is 0). The new size is clipped to be at least the current stack size and at most `parisizemax`. The stack content is not affected by this operation.

`void parivstack_reset(void)` resets the current stack to its default size `parisize`. This is used to recover memory after a computation that enlarged the stack. This function destroys the content of the enlarged stack (between the old and the new bottom of the stack). Before calling this function, you must ensure that `avma` lies within the new smaller stack.

`void paristack_newsize(ulong newsize)` (*does not return*). Library version of
`default(parisize, "newsize")`

Set the default `parisize` to `newsize`, or double `parisize` if `newsize` is equal to 0, then call `cb_pari_err_recover(-1)`.

`void parivstack_resize(ulong newsize)` (*does not return*). Library version of
`default(parisizemax, "newsize")`

Set the default `parisizemax` to `newsize` and call `cb_pari_err_recover(-1)`.

5.1.3 Notions specific to the GP interpreter.

An **entree** is the generic object attached to an identifier (a name) in GP's interpreter, be it a built-in or user function, or a variable. For a function, it has at least the following fields:

`char *name`: the name under which the interpreter knows us.

`void *value`: a pointer to the C function to call.

`long menu`: a small integer ≥ 1 (to which group of function help do we belong, for the `?n` help menu).

`char *code`: the prototype code.

`char *help`: the help text for the function.

A routine in GP is described to the analyzer by an **entree** structure. Built-in PARI routines are grouped in *modules*, which are arrays of **entree** structs, the last of which satisfy `name = NULL` (sentinel). There are currently four modules in PARI/GP:

- general functions (`functions_basic`, known to `libpari`),
- gp-specific functions (`functions_gp`),

and two modules of obsolete functions. The function `pari_init` initializes the interpreter and declares all symbols in `functions_basic`. You may declare further functions on a case by case basis or as a whole module using

`void pari_add_function(entree *ep)` adds a single routine to the table of symbols in the interpreter. It assumes `pari_init` has been called.

`void pari_add_module(entree *mod)` adds all the routines in module `mod` to the table of symbols in the interpreter. It assumes `pari_init` has been called.

For instance, `gp` implements a number of private routines, which it adds to the default set via the calls

```
pari_add_module(functions_gp);
```

A GP `default` is likewise attached to a helper routine, that is run when the value is consulted, or changed by `default0` or `setdefault`. Such routines are grouped in the module `functions_default`.

`void pari_add_defaults_module(entree *mod)` adds all the defaults in module `mod` to the interpreter. It assumes that `pari_init` has been called. From this point on, all defaults in module `mod` are known to `setdefault` and friends.

5.1.4 Public callbacks.

The `gp` calculator associates elaborate functions (for instance the break loop handler) to the following callbacks, and so can you:

`void (*cb_pari_ask_confirm)(const char *s)` initialized to `NULL`. Called with argument `s` whenever PARI wants confirmation for action `s`, for instance in `secure` mode.

`void (*cb_pari_init_histfile)(void)` initialized to `NULL`. Called when the `histfile` default is changed. The intent is for that callback to read the file content, append it to history in memory, then dump the expanded history to the new `histfile`.

`int (*cb_pari_is_interactive)(void)`; initialized to `NULL`.

`void (*cb_pari_quit)(long)` initialized to a no-op. Called when `gp` must evaluate the `quit` command.

`void (*cb_pari_start_output)(void)` initialized to `NULL`.

`int (*cb_pari_handle_exception)(long)` initialized to `NULL`. If not `NULL`, this routine is called with argument `-1` on `SIGINT`, and argument `err` on error `err`. If it returns a non-zero value, the error or signal handler returns, in effect further ignoring the error or signal, otherwise it raises a fatal error. A possible simple-minded handler, used by the `gp` interpreter, is

`int gp_handle_exception(long err)` if the `breakloop` default is enabled (set to 1) and `cb_pari_break_loop` is not `NULL`, we call this routine with `err` argument and return the result.

`int (*cb_pari_err_handle)(GEN)` If not `NULL`, this routine is called with a `t_ERROR` argument from `pari_err`. If it returns a non-zero value, the error returns, in effect further ignoring the error, otherwise it raises a fatal error.

The default behavior is to print a descriptive error message (display the error), then return 0, thereby raising a fatal error. This differs from `cb_pari_handle_exception` in that the function is not called on `SIGINT` (which do not generate a `t_ERROR`), only from `pari_err`. Use `cb_pari_sigint` if you need to handle `SIGINT` as well.

The following function can be used by `cb_pari_err_handle` to display the error message.

`const char* closure_func_err()` return a statically allocated string holding the name of the function that triggered the error. Return `NULL` if the error was not caused by a function.

`int (*cb_pari_break_loop)(int)` initialized to `NULL`.

`void (*cb_pari_sigint)(void)`. Function called when we receive `SIGINT`. By default, raises

```
pari_err(e_MISC, "user interrupt");
```

A possible simple-minded variant, used by the `gp` interpreter, is

```
void gp_sigint_fun(void)
```

`void (*cb_pari_pre_recover)(long)` initialized to `NULL`. If not `NULL`, this routine is called just before PARI cleans up from an error. It is not required to return. The error number is passed as argument, unless the PARI stack has been destroyed (`allocatemem`), in which case `-1` is passed.

`void (*cb_pari_err_recover)(long)` initialized to `pari_exit()`. This callback must not return. It is called after PARI has cleaned-up from an error. The error number is passed as argument, unless the PARI stack has been destroyed, in which case it is called with argument `-1`.

`int (*cb_pari_whatnow)(PariOUT *out, const char *s, int flag)` initialized to `NULL`. If not `NULL`, must check whether `s` existed in older versions of `pari` (the `gp` callback checks against `pari-1.39.15`). All output must be done via `out` methods.

- `flag = 0`: should print verbosely the answer, including help text if available.
- `flag = 1`: must return 0 if the function did not change, and a non-0 result otherwise. May print a help message.

5.1.5 Configuration variables.

`pari_library_path`: If set, It should be a path to the `libpari` library. It is used by the function `gpinstall` to locate the PARI library when searching for symbols. This should only be useful on Windows.

5.1.6 Utility functions.

`void pari_ask_confirm(const char *s)` raise an error if the callback `cb_pari_ask_confirm` is `NULL`. Otherwise calls

```
cb_pari_ask_confirm(s);
```

`char* gp_filter(const char *s)` pre-processor for the GP parser: filter out whitespace and GP comments from `s`.

GEN `pari_compile_str(const char *s)` low-level form of `compile_str`: assumes that `s` does not contain spaces or GP comments and returns the closure attached to the GP expression `s`. Note that GP metacommands are not recognized.

`int gp_meta(const char *s, int ismain)` low-level component of `gp_read_str`: assumes that `s` does not contain spaces or GP comments and try to interpret `s` as a GP metacommand (e.g. starting by `\` or `?`). If successful, execute the metacommand and return 1; otherwise return 0. The `ismain` parameter modifies the way `\r` commands are handled: if non-zero, act as if the file contents were entered via standard input (i.e. call `switchin` and divert `pari_infile`); otherwise, simply call `gp_read_file`.

`void pari_hit_return(void)` wait for the use to enter `\n` via standard input.

`void gp_load_gprc(void)` read and execute the user's GPRC file.

`void pari_center(const char *s)` print `s`, centered.

`void pari_print_version(void)` print verbose version information.

`long pari_community(void)` return the index of the support section n the help.

`const char* gp_format_time(long t)` format a delay of `t` ms suitable for `gp` output, with timer set.

`const char* gp_format_prompt(const char *p)` format a prompt `p` suitable for `gp` prompting (includes colors and protecting ANSI escape sequences for readline).

`void pari_alarm(long s)` set an alarm after `s` seconds (raise an `e_ALARM` exception).

`void gp_help(const char *s, long flag)` print help for *s*, depending on the value of *flag*:

- `h_REGULAR`, basic help (?);
- `h_LONG`, extended help (??);
- `h_APROPOS`, a propos help (??).

`const char ** gphelp_keyword_list(void)` return a NULL-terminated array a strings, containing keywords known to `gphelp` besides GP functions (e.g. `modulus` or `operator`). Used by the online help system and the contextual completion engine.

`void gp_echo_and_log(const char *p, const char *s)` given a prompt *p* and attached input command *s*, update logfile and possibly print on standard output if `echo` is set and we are not in interactive mode. The callback `cb_pari_is_interactive` must be set to a sensible value.

`void gp_alarm_handler(int sig)` the SIGALRM handler set by the `gp` interpreter.

`void print_fun_list(char **list, long n)` print all elements of `list` in columns, pausing (hit return) every *n* lines. `list` is NULL terminated.

5.1.7 Saving and restoring the GP context.

`void gp_context_save(struct gp_context* rec)` save the current GP context.

`void gp_context_restore(struct gp_context* rec)` restore a GP context. The new context must be an ancestor of the current context.

5.1.8 GP history.

These functions allow to control the GP history (the `%` operator).

`void pari_add_hist(GEN x, long t)` adds *x* as the last history entry; *t* is the time we used to compute it.

`GEN pari_get_hist(long p)`, if *p* > 0 returns entry of index *p* (i.e. `%p`), else returns entry of index *n* + *p* where *n* is the index of the last entry (used for `%`, `%'`, `%''`, etc.).

`long pari_get_histtime(long p)` as `pari_get_hist`, returning the time used to compute the history entry, instead of the entry itself.

`ulong pari_nb_hist(void)` return the index of the last entry.

5.2 Handling GENs.

Almost all these functions are either macros or inlined. Unless mentioned otherwise, they do not evaluate their arguments twice. Most of them are specific to a set of types, although no consistency checks are made: e.g. one may access the `sign` of a `t_PADIC`, but the result is meaningless.

5.2.1 Allocation.

GEN `cgetg(long l, long t)` allocates (the root of) a GEN of type t and length l . Sets $z[0]$.

GEN `cgeti(long l)` allocates a `t_INT` of length l (including the 2 codewords). Sets $z[0]$ only.

GEN `cgetr(long l)` allocates a `t_REAL` of length l (including the 2 codewords). Sets $z[0]$ only.

GEN `cgetc(long prec)` allocates a `t_COMPLEX` whose real and imaginary parts are `t_REALs` of length `prec`.

GEN `cgetg_copy(GEN x, long *lx)` fast version of `cgetg`: allocate a GEN with the same type and length as x , setting `*lx` to `lg(x)` as a side-effect. (Only sets the first codeword.) This is a little faster than `cgetg` since we may reuse the bitmask in $x[0]$ instead of recomputing it, and we do not need to check that the length does not overflow the possibilities of the implementation (since an object with that length already exists). Note that `cgetg` with arguments known at compile time, as in

```
cgetg(3, t_INTMOD)
```

will be even faster since the compiler will directly perform all computations and checks.

GEN `vec trunc_init(long l)` perform `cgetg(1,t_VEC)`, then set the length to 1 and return the result. This is used to implement vectors whose final length is easily bounded at creation time, that we intend to fill gradually using:

`void vec trunc_append(GEN x, GEN y)` assuming x was allocated using `vec trunc_init`, appends y as the last element of x , which grows in the process. The function is shallow: we append y , not a copy; it is equivalent to

```
long lx = lg(x); gel(x,lx) = y; setlg(x, lx+1);
```

Beware that the maximal size of x (the l argument to `vec trunc_init`) is unknown, hence unchecked, and stack corruption will occur if we append more than $l - 1$ elements to x . Use the safer (but slower) `shallowconcat` when l is not easy to bound in advance.

An other possibility is simply to allocate using `cgetg(1, t)` then fill the components as they become available: this time the downside is that we do not obtain a correct GEN until the vector is complete. Almost no PARI function will be able to operate on it.

`void vec trunc_append_batch(GEN x, GEN y)` successively apply

```
vec trunc_append(x, gel(y, i))
```

for all elements of the vector y .

GEN `col trunc_init(long l)` as `vec trunc_init` but perform `cgetg(1,t_COL)`.

GEN `vec small trunc_init(long l)`

`void vec small trunc_append(GEN x, long t)` analog to the above for a `t_VECSMALL` container.

5.2.2 Length conversions.

These routines convert a non-negative length to different units. Their behavior is undefined at negative integers.

`long ndec2nlong(long x)` converts a number of decimal digits to a number of words. Returns $1 + \text{floor}(x \times \text{BITS_IN_LONG} \log_2 10)$.

`long ndec2prec(long x)` converts a number of decimal digits to a number of codewords. This is equal to $2 + \text{ndec2nlong}(x)$.

`long ndec2nbits(long x)` converts a number of decimal digits to a number of bits.

`long prec2ndec(long x)` converts a number of codewords to a number of decimal digits.

`long nbits2nlong(long x)` converts a number of bits to a number of words. Returns the smallest word count containing x bits, i.e. $\text{ceil}(x/\text{BITS_IN_LONG})$.

`long nbits2ndec(long x)` converts a number of bits to a number of decimal digits.

`long nbits2lg(long x)` converts a number of bits to a length in code words. Currently an alias for `nbits2nlong`.

`long nbits2prec(long x)` converts a number of bits to a number of codewords. This is equal to $2 + \text{nbits2nlong}(x)$.

`long nbits2extraprec(long x)` converts a number of bits to the mantissa length of a `t_REAL` in codewords. This is currently an alias to `nbits2nlong(x)`.

`long nchar2nlong(long x)` converts a number of bytes to number of words. Returns the smallest word count containing x bytes, i.e. $\text{ceil}(x/\text{sizeof}(\text{long}))$.

`long prec2nbits(long x)` converts a `t_REAL` length into a number of significant bits; returns $(x - 2)\text{BITS_IN_LONG}$.

`double prec2nbits_mul(long x, double y)` returns $\text{prec2nbits}(x) \times y$.

`long bit_accuracy(long x)` converts a length into a number of significant bits; currently an alias for `prec2nbits`.

`double bit_accuracy_mul(long x, double y)` returns $\text{bit_accuracy}(x) \times y$.

`long realprec(GEN x)` length of a `t_REAL` in words; currently an alias for `lg`.

`long bit_prec(GEN x)` length of a `t_REAL` in bits.

`long precdbl(long prec)` given a length in words corresponding to a `t_REAL` precision, return the length corresponding to doubling the precision. Due to the presence of 2 code words, this is $2(\text{prec} - 2) + 2$.

5.2.3 Read type-dependent information.

`long typ(GEN x)` returns the type number of x . The header files included through `pari.h` define symbolic constants for the GEN types: `t_INT` etc. Never use their actual numerical values. E.g to determine whether x is a `t_INT`, simply check

```
if (typ(x) == t_INT) { }
```

The types are internally ordered and this simplifies the implementation of commutative binary operations (e.g addition, gcd). Avoid using the ordering directly, as it may change in the future; use type grouping functions instead (Section 5.2.6).

`const char* type_name(long t)` given a type number t this routine returns a string containing its symbolic name. E.g `type_name(t_INT)` returns `"t_INT"`. The return value is read-only.

`long lg(GEN x)` returns the length of x in `BITS_IN_LONG`-bit words.

`long lgfint(GEN x)` returns the effective length of the `t_INT` x in `BITS_IN_LONG`-bit words.

`long signe(GEN x)` returns the sign (-1 , 0 or 1) of x . Can be used for `t_INT`, `t_REAL`, `t_POL` and `t_SER` (for the last two types, only 0 or 1 are possible).

`long gsigne(GEN x)` returns the sign of a real number x , valid for `t_INT`, `t_REAL` as `signe`, but also for `t_FRAC` and `t_QUAD` of positive discriminants. Raise a type error if `typ(x)` is not among those.

`long expi(GEN x)` returns the binary exponent of the real number equal to the `t_INT` x . This is a special case of `gexpo`.

`long expo(GEN x)` returns the binary exponent of the `t_REAL` x .

`long mpexpo(GEN x)` returns the binary exponent of the `t_INT` or `t_REAL` x .

`long gexpo(GEN x)` same as `expo`, but also valid when x is not a `t_REAL` (returns the largest exponent found among the components of x). When x is an exact 0 , this returns `-HIGHEXPOBIT`, which is lower than any valid exponent.

`long gexpo_safe(GEN x)` same as `gexpo`, but returns a value strictly less than `-HIGHEXPOBIT` when the exponent is not defined (e.g. for a `t_PADIC` or `t_INTMOD` component).

`long valp(GEN x)` returns the p -adic valuation (for a `t_PADIC`) or X -adic valuation (for a `t_SER`, taken with respect to the main variable) of x .

`long precp(GEN x)` returns the precision of the `t_PADIC` x .

`long varn(GEN x)` returns the variable number of the `t_POL` or `t_SER` x (between 0 and `MAXVARN`).

`long gvar(GEN x)` returns the main variable number when any variable at all occurs in the composite object x (the smallest variable number which occurs), and `NO_VARIABLE` otherwise.

`long gvar2(GEN x)` returns the variable number for the ring over which x is defined, e.g. if $x \in \mathbf{Z}[a][b]$ return (the variable number for) a . Return `NO_VARIABLE` if x has no variable or is not defined over a polynomial ring.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL` x with respect to its main variable, *if* its leading coefficient is non-zero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0 , e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1 .

`long lgpol(GEN x)` is equal to `degpol(x) + 1`. Used to loop over the coefficients of a `t_POL` in the following situation:

```
GEN xd = x + 2;
long i, l = lgpol(x);
for (i = 0; i < l; i++) foo( xd[i] ).
```

`long precision(GEN x)` If `x` is of type `t_REAL`, returns the precision of `x`, namely the length of `x` in `BITS_IN_LONG`-bit words if `x` is not zero, and a reasonable quantity obtained from the exponent of `x` if `x` is numerically equal to zero. If `x` is of type `t_COMPLEX`, returns the minimum of the precisions of the real and imaginary part. Otherwise, returns 0 (which stands for infinite precision).

`long lgcols(GEN x)` is equal to `lg(gel(x,1))`. This is the length of the columns of a `t_MAT` with at least one column.

`long nbrows(GEN x)` is equal to `lg(gel(x,1))-1`. This is the number of rows of a `t_MAT` with at least one column.

`long gprecision(GEN x)` as `precision` for scalars. Returns the lowest precision encountered among the components otherwise.

`long sizedigit(GEN x)` returns 0 if `x` is exactly 0. Otherwise, returns `gexpo(x)` multiplied by $\log_{10}(2)$. This gives a crude estimate for the maximal number of decimal digits of the components of `x`.

5.2.4 Eval type-dependent information. These routines convert type-dependent information to bitmask to fill the codewords of `GEN` objects (see Section 4.5). E.g for a `t_REAL` `z`:

```
z[1] = evalsigne(-1) | evalexpo(2)
```

Compatible components of a codeword for a given type can be OR-ed as above.

`ulong evaltyp(long x)` convert type `x` to bitmask (first codeword of all `GENs`)

`long evallg(long x)` convert length `x` to bitmask (first codeword of all `GENs`). Raise overflow error if `x` is so large that the corresponding length cannot be represented

`long _evallg(long x)` as `evallg` *without* the overflow check.

`ulong evalvarn(long x)` convert variable number `x` to bitmask (second codeword of `t_POL` and `t_SER`)

`long evalsigne(long x)` convert sign `x` (in $-1, 0, 1$) to bitmask (second codeword of `t_INT`, `t_REAL`, `t_POL`, `t_SER`)

`long evalprecp(long x)` convert p -adic (X -adic) precision `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding precision cannot be represented.

`long _evalprecp(long x)` same as `evalprecp` *without* the overflow check.

`long evalvalp(long x)` convert p -adic (X -adic) valuation `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding valuation cannot be represented.

`long _evalvalp(long x)` same as `evalvalp` *without* the overflow check.

`long evalexpo(long x)` convert exponent x to bitmask (second codeword of `t_REAL`). Raise overflow error if x is so large that the corresponding exponent cannot be represented

`long _evalexpo(long x)` same as `evalexpo` *without* the overflow check.

`long evallgefint(long x)` convert effective length x to bitmask (second codeword `t_INT`). This should be less or equal than the length of the `t_INT`, hence there is no overflow check for the effective length.

5.2.5 Set type-dependent information. Use these functions and macros with extreme care since usually the corresponding information is set otherwise, and the components and further codeword fields (which are left unchanged) may not be compatible with the new information.

`void settyp(GEN x, long s)` sets the type number of x to s .

`void setlg(GEN x, long s)` sets the length of x to s . This is an efficient way of truncating vectors, matrices or polynomials.

`void setlgefint(GEN x, long s)` sets the effective length of the `t_INT` x to s . The number s must be less than or equal to the length of x .

`void setsigne(GEN x, long s)` sets the sign of x to s . If x is a `t_INT` or `t_REAL`, s must be equal to -1 , 0 or 1 , and if x is a `t_POL` or `t_SER`, s must be equal to 0 or 1 . No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void togglesign(GEN x)` sets the sign s of x to $-s$, in place.

`void togglesign_safe(GEN *x)` sets the s sign of $*x$ to $-s$, in place, unless $*x$ is one of the integer universal constants in which case replace $*x$ by its negation (e.g. replace `gen_1` by `gen_m1`).

`void setabssign(GEN x)` sets the sign s of x to $|s|$, in place.

`void affectsign(GEN x, GEN y)` shortcut for `setsigne(y, signe(x))`. No sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void affectsign_safe(GEN x, GEN *y)` sets the sign of $*y$ to that of x , in place, unless $*y$ is one of the integer universal constants in which case replace $*y$ by its negation if needed (e.g. replace `gen_1` by `gen_m1` if x is negative). No other sanity check is made; in particular, setting the sign of a 0 `t_INT` to ± 1 creates an invalid object.

`void normalize_frac(GEN z)` assuming z is of the form `mkfrac(a,b)` with $b \neq 0$, make sure that $b > 0$ by changing the sign of a in place if needed (use `togglesign`).

`void setexpo(GEN x, long s)` sets the binary exponent of the `t_REAL` x to s . The value s must be a 24-bit signed number.

`void setvalp(GEN x, long s)` sets the p -adic or X -adic valuation of x to s , if x is a `t_PADIC` or a `t_SER`, respectively.

`void setprec(GEN x, long s)` sets the p -adic precision of the `t_PADIC` x to s .

`void setvarn(GEN x, long s)` sets the variable number of the `t_POL` or `t_SER` x to s (where $0 \leq s \leq \text{MAXVARN}$).

5.2.6 Type groups. In the following functions, `t` denotes the type of a `GEN`. They used to be implemented as macros, which could evaluate their argument twice; *no longer*: it is not inefficient to write

```
is_intreal_t(typ(x))
```

`int is_recursive_t(long t)` true iff `t` is a recursive type (the non-recursive types are `t_INT`, `t_REAL`, `t_STR`, `t_VECSMALL`). Somewhat contrary to intuition, `t_LIST` is also non-recursive, ; see the Developer's guide for details.

`int is_intreal_t(long t)` true iff `t` is `t_INT` or `t_REAL`.

`int is_rational_t(long t)` true iff `t` is `t_INT` or `t_FRAC`.

`int is_real_t(long t)` true iff `t` is `t_INT` or `t_REAL` or `t_FRAC`.

`int is_vec_t(long t)` true iff `t` is `t_VEC` or `t_COL`.

`int is_matvec_t(long t)` true iff `t` is `t_MAT`, `t_VEC` or `t_COL`.

`int is_scalar_t(long t)` true iff `t` is a scalar, i.e a `t_INT`, a `t_REAL`, a `t_INTMOD`, a `t_FRAC`, a `t_COMPLEX`, a `t_PADIC`, a `t_QUAD`, or a `t_POLMOD`.

`int is_extscalar_t(long t)` true iff `t` is a scalar (see `is_scalar_t`) or `t` is `t_POL`.

`int is_const_t(long t)` true iff `t` is a scalar which is not `t_POLMOD`.

`int is_noncalc_t(long t)` true if generic operations (`gadd`, `gmul`) do not make sense for `t`: corresponds to types `t_LIST`, `t_STR`, `t_VECSMALL`, `t_CLOSURE`

5.2.7 Accessors and components. The first two functions return `GEN` components as copies on the stack:

`GEN compo(GEN x, long n)` creates a copy of the `n`-th true component (i.e. not counting the codewords) of the object `x`.

`GEN truecoeff(GEN x, long n)` creates a copy of the coefficient of degree `n` of `x` if `x` is a scalar, `t_POL` or `t_SER`, and otherwise of the `n`-th component of `x`.

On the contrary, the following routines return the address of a `GEN` component. No copy is made on the stack:

`GEN constant_coeff(GEN x)` returns the address of the constant coefficient of `t_POL` `x`. By convention, a 0 polynomial (whose `sign` is 0) has `gen_0` constant term.

`GEN leading_coeff(GEN x)` returns the address of the leading coefficient of `t_POL` `x`, i.e. the coefficient of largest index stored in the array representing `x`. This may be an inexact 0. By convention, return `gen_0` if the coefficient array is empty.

`GEN gel(GEN x, long i)` returns the address of the `x[i]` entry of `x`. (`e1` stands for element.)

`GEN gcoeff(GEN x, long i, long j)` returns the address of the `x[i,j]` entry of `t_MAT` `x`, i.e. the coefficient at row `i` and column `j`.

`GEN gmael(GEN x, long i, long j)` returns the address of the `x[i][j]` entry of `x`. (`mael` stands for multidimensional array element.)

`GEN gmael2(GEN A, long x1, long x2)` is an alias for `gmael`. Similar macros `gmael3`, `gmael4`, `gmael5` are available.

5.3 Global numerical constants.

These are defined in the various public PARI headers.

5.3.1 Constants related to word size.

`long BITS_IN_LONG = 2TWOPOTBITS_IN_LONG`: number of bits in a `long` (32 or 64).

`long BITS_IN_HALFULONG`: `BITS_IN_LONG` divided by 2.

`long LONG_MAX`: the largest positive `long`.

`ulong ULONG_MAX`: the largest `ulong`.

`long DEFAULTPREC`: the length (`lg`) of a `t_REAL` with 64 bits of accuracy

`long MEDDEFAULTPREC`: the length (`lg`) of a `t_REAL` with 128 bits of accuracy

`long BIGDEFAULTPREC`: the length (`lg`) of a `t_REAL` with 192 bits of accuracy

`ulong HIGHBIT`: the largest power of 2 fitting in an `ulong`.

`ulong LOWMASK`: bitmask yielding the least significant bits.

`ulong HIGHMASK`: bitmask yielding the most significant bits.

The last two are used to implement the following convenience macros, returning half the bits of their operand:

`ulong LOWWORD(ulong a)` returns least significant bits.

`ulong HIGHWORD(ulong a)` returns most significant bits.

Finally

`long divsBIL(long n)` returns the Euclidean quotient of n by `BITS_IN_LONG` (with non-negative remainder).

`long remsBIL(n)` returns the (non-negative) Euclidean remainder of n by `BITS_IN_LONG`

`long dvmdsBIL(long n, long *r)`

`ulong dvmduBIL(ulong n, ulong *r)` sets r to `remsBIL(n)` and returns `divsBIL(n)`.

5.3.2 Masks used to implement the GEN type.

These constants are used by higher level macros, like `typ` or `lg`:

`EXPOnumBITS`, `LGnumBITS`, `SIGNnumBITS`, `TYPnumBITS`, `VALPnumBITS`, `VARNnumBITS`: number of bits used to encode `expo`, `lg`, `signe`, `typ`, `valp`, `varn`.

`PRECPSHIFT`, `SIGNSHIFT`, `TYPSHIFT`, `VARNSHIFT`: shifts used to recover or encode `precp`, `varn`, `typ`, `signe`

`CLONEBIT`, `EXPOBITS`, `LGBITS`, `PRECPBITS`, `SIGNBITS`, `TYPBITS`, `VALPBITS`, `VARNBITS`: bitmasks used to extract `isclone`, `expo`, `lg`, `precp`, `signe`, `typ`, `valp`, `varn` from `GEN` codewords.

`MAXVARN`: the largest possible variable number.

`NO_VARIABLE`: sentinel returned by `gvar(x)` when x does not contain any polynomial; has a lower priority than any valid variable number.

`HIGHEXPOBIT`: a power of 2, one more than the largest possible exponent for a `t_REAL`.

`HIGHVALPBIT`: a power of 2, one more than the largest possible valuation for a `t_PADIC` or a `t_SER`.

5.3.3 $\log 2$, π .

These are double approximations to useful constants:

M_PI: π .

M_LN2: $\log 2$.

LOG10_2: $\log 2 / \log 10$.

LOG2_10: $\log 10 / \log 2$.

5.4 Iterating over small primes, low-level interface.

One of the methods used by the high-level prime iterator (see Section 4.8.2), is a precomputed table. Its direct use is deprecated, but documented here.

After `pari_init(size, maxprime)`, a “prime table” is initialized with the successive *differences* of primes up to (possibly just a little beyond) `maxprime`. The prime table occupies roughly `maxprime / log(maxprime)` bytes in memory, so be sensible when choosing `maxprime`; it is 500000 by default under `gp` and there is no real benefit in choosing a much larger value: the high-level iterator provide *fast* access to primes up to the *square* of `maxprime`. In any case, the implementation requires that `maxprime < 2BITS_IN_LONG - 2048`, whatever memory is available.

PARI currently guarantees that the first 6547 primes, up to and including 65557, are present in the table, even if you set `maxprime` to zero. in the `pari_init` call.

Some convenience functions:

`ulong maxprime()` the largest prime computable using our prime table.

`void maxprime_check(ulong B)` raise an error if `maxprime()` is $< B$.

After the following initializations (the names `p` and `ptr` are arbitrary of course)

```
byteptr ptr = diffptr;
ulong p = 0;
```

calling the macro `NEXT_PRIME_VIADIFF_CHECK(p, ptr)` repeatedly will assign the successive prime numbers to `p`. Overrunning the prime table boundary will raise the error `e_MAXPRIME`, which just prints the error message:

```
*** not enough precomputed primes, need primelimit ~c
```

(for some numerical value c), then the macro aborts the computation. The alternative macro `NEXT_PRIME_VIADIFF` operates in the same way, but will omit that check, and is slightly faster. It should be used in the following way:

```
byteptr ptr = diffptr;
ulong p = 0;

if (maxprime() < goal) pari_err_MAXPRIME(goal); /* not enough primes */
while (p <= goal) /* run through all primes up to goal */
{
    NEXT_PRIME_VIADIFF(p, ptr);
    ...
}
```

Here, we use the general error handling function `pari_err` (see Section 4.7.3), with the codeword `e_MAXPRIME`, raising the “not enough primes” error. This could be rewritten as

```

maxprime_check(goal);
while (p <= goal) /* run through all primes up to goal */
{
    NEXT_PRIME_VIADIFF(p, ptr);
    ...
}

```

`byteptr initprimes(ulong maxprime, long *L, ulong *lastp)` computes a (malloc’ed) “prime table”, in fact a table of all prime differences for $p < \text{maxprime}$ (and possibly a little beyond). Set L to the table length (argument to `malloc`), and $lastp$ to the last prime in the table.

`void initprimetable(ulong maxprime)` computes a prime table (of all prime differences for $p < \text{maxprime}$) and assign it to the global variable `diffptr`. Don’t change `diffptr` directly, call this function instead. This calls `initprimes` and updates internal data recording the table size.

`ulong init_primepointer_geq(ulong a, byteptr *pd)` returns the smallest prime $p \geq a$, and sets `*pd` to the proper offset of `diffptr` so that `NEXT_PRIME_VIADIFF(p, *pd)` correctly returns `unextprime(p + 1)`.

`ulong init_primepointer_gt(ulong a, byteptr *pd)` returns the smallest prime $p > a$.

`ulong init_primepointer_leq(ulong a, byteptr *pd)` returns the largest prime $p \leq a$.

`ulong init_primepointer_lt(ulong a, byteptr *pd)` returns the largest prime $p < a$.

5.5 Handling the PARI stack.

5.5.1 Allocating memory on the stack.

`GEN cgetg(long n, long t)` allocates memory on the stack for an object of length n and type t , and initializes its first codeword.

`GEN cgeti(long n)` allocates memory on the stack for a `t_INT` of length n , and initializes its first codeword. Identical to `cgetg(n, t_INT)`.

`GEN cgetr(long n)` allocates memory on the stack for a `t_REAL` of length n , and initializes its first codeword. Identical to `cgetg(n, t_REAL)`.

`GEN cgetc(long n)` allocates memory on the stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REALs` of length n .

`GEN cgetp(GEN x)` creates space sufficient to hold the `t_PADIC` x , and sets the prime p and the p -adic precision to those of x , but does not copy (the p -adic unit or zero representative and the modulus of) x .

`GEN new_chunk(size_t n)` allocates a `GEN` with n components, *without* filling the required code words. This is the low-level constructor underlying `cgetg`, which calls `new_chunk` then sets the first code word. It works by simply returning the address `((GEN)avma) - n`, after checking that it is larger than `(GEN)bot`.

`void new_chunk_resize(size_t x)` this function is called by `new_chunk` when the PARI stack overflows. There is no need to call it manually. It will either extend the stack or report an `e_STACK` error.

`char* stack_malloc(size_t n)` allocates memory on the stack for n chars (*not* n GENs). This is faster than using `malloc`, and easier to use in most situations when temporary storage is needed. In particular there is no need to `free` individually all variables thus allocated: a simple `avma = oldavma` might be enough. On the other hand, beware that this is not permanent independent storage, but part of the stack. The memory is aligned on `sizeof(long)` bytes boundaries.

`char* stack_malloc_align(size_t n, long k)` as `stack_malloc`, but the memory is aligned on k bytes boundaries. The number k must be a multiple of the `sizeof(long)`.

`char* stack_calloc(size_t n)` as `stack_malloc`, setting the memory to zero.

Objects allocated through these last three functions cannot be `gerepile`'d, since they are not yet valid GENs: their codewords must be filled first.

GEN `cgetalloc(long t, size_t l)`, same as `cgetg(t, l)`, except that the result is allocated using `pari_malloc` instead of the PARI stack. The resulting GEN is now impervious to garbage collecting routines, but should be freed using `pari_free`.

5.5.2 Stack-independent binary objects.

GENbin* `copy_bin(GEN x)` copies x into a malloc'ed structure suitable for stack-independent binary transmission or storage. The object obtained is architecture independent provided, `sizeof(long)` remains the same on all PARI instances involved, as well as the multiprecision kernel (either native or GMP).

GENbin* `copy_bin_canon(GEN x)` as `copy_bin`, ensuring furthermore that the binary object is independent of the multiprecision kernel. Slower than `copy_bin`.

GEN `bin_copy(GENbin *p)` assuming p was created by `copy_bin(x)` (not necessarily by the same PARI instance: transmission or external storage may be involved), restores x on the PARI stack.

The routine `bin_copy` transparently encapsulate the following functions:

GEN `GENbinbase(GENbin *p)` the GEN data actually stored in p . All addresses are stored as offsets with respect to a common reference point, so the resulting GEN is unusable unless it is a non-recursive type; private low-level routines must be called first to restore absolute addresses.

void `shiftaddress(GEN x, long dec)` converts relative addresses to absolute ones.

void `shiftaddress_canon(GEN x, long dec)` converts relative addresses to absolute ones, and converts leaves from a canonical form to the one specific to the multiprecision kernel in use. The GENbin type stores whether leaves are stored in canonical form, so `bin_copy` can call the right variant.

Objects containing closures are harder to e.g. copy and save to disk, since closures contain pointers to libpari functions that will not be valid in another gp instance: there is little chance for them to be loaded at the exact same address in memory. Such objects must be saved along with a linking table.

GEN `copybin_unlink(GEN C)` returns a linking table allowing to safely store and transmit `t_CLOSURE` objects in C . If $C = \text{NULL}$ return a linking table corresponding to the content of all gp variables. C may then be dumped to disk in binary form, for instance.

void `bincopy_relink(GEN C, GEN V)` given a binary object C , as dumped by `writebin` and read back into a session, and a linking table V , restore all closures contained in C (function pointers are translated to their current value).

5.5.3 Garbage collection. See Section 4.3 for a detailed explanation and many examples.

`void cgiv(GEN x)` frees object `x`, assuming it is the last created on the stack.

`GEN gerepile(pari_sp p, pari_sp q, GEN x)` general garbage collector for the stack.

`void gerepileall(pari_sp av, int n, ...)` cleans up the stack from `av` on (i.e from `avma` to `av`), preserving the `n` objects which follow in the argument list (of type `GEN*`). For instance, `gerepileall(av, 2, &x, &y)` preserves `x` and `y`.

`void gerepileallsp(pari_sp av, pari_sp ltop, int n, ...)` cleans up the stack between `av` and `ltop`, updating the `n` elements which follow `n` in the argument list (of type `GEN*`). Check that the elements of `g` have no component between `av` and `ltop`, and assumes that no garbage is present between `avma` and `ltop`. Analogous to (but faster than) `gerepileall` otherwise.

`GEN gerepilecopy(pari_sp av, GEN x)` cleans up the stack from `av` on, preserving the object `x`. Special case of `gerepileall` (case `n = 1`), except that the routine returns the preserved `GEN` instead of updating its address through a pointer.

`void gerepilemany(pari_sp av, GEN* g[], int n)` alternative interface to `gerepileall`. The preserved `GENs` are the elements of the array `g` of length `n`: `g[0], g[1], ..., g[n-1]`. Obsolete: no more efficient than `gerepileall`, error-prone, and clumsy (need to declare an extra `GEN *g`).

`void gerepilemanysp(pari_sp av, pari_sp ltop, GEN* g[], int n)` alternative interface to `gerepileallsp`. Obsolete.

`void gerepilecoeffs(pari_sp av, GEN x, int n)` cleans up the stack from `av` on, preserving `x[0], ..., x[n-1]` (which are `GENs`).

`void gerepilecoeffssp(pari_sp av, pari_sp ltop, GEN x, int n)` cleans up the stack from `av` to `ltop`, preserving `x[0], ..., x[n-1]` (which are `GENs`). Same assumptions as in `gerepilemanysp`, of which this is a variant. For instance

```
z = cgetg(3, t_COMPLEX);
av = avma; garbage(); ltop = avma;
z[1] = fun1();
z[2] = fun2();
gerepilecoeffssp(av, ltop, z + 1, 2);
return z;
```

cleans up the garbage between `av` and `ltop`, and connects `z` and its two components. This is marginally more efficient than the standard

```
av = avma; garbage(); ltop = avma;
z = cgetg(3, t_COMPLEX);
z[1] = fun1();
z[2] = fun2(); return gerepile(av, ltop, z);
```

`GEN gerepileupto(pari_sp av, GEN q)` analogous to (but faster than) `gerepilecopy`. Assumes that `q` is connected and that its root was created before any component. If `q` is not on the stack, this is equivalent to `avma = av`; in particular, sentinels which are not even proper `GENs` such as `q = NULL` are allowed.

`GEN gerepileuptoint(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a `t_INT`. The length and effective length of the resulting `t_INT` are equal.

GEN `gerepileuptoleaf(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a leaf, i.e a non-recursive type (`is_recursive_t(typ(q))` is non-zero). Contrary to `gerepileuptoint` and `gerepileupto`, `gerepileuptoleaf` leaves length and effective length of a `t_INT` unchanged.

5.5.4 Garbage collection: advanced use.

`void stackdummy(pari_sp av, pari_sp ltop)` inhibits the memory area between `av` *included* and `ltop` *excluded* with respect to `gerepile`, in order to avoid a call to `gerepile(av, ltop, ...)`. The stack space is not reclaimed though.

More precisely, this routine assumes that `av` is recorded earlier than `ltop`, then marks the specified stack segment as a non-recursive type of the correct length. Thus `gerepile` will not inspect the zone, at most copy it. To be used in the following situation:

```
av0 = avma; z = cgetg(t_VEC, 3);
gel(z,1) = HUGE(); av = avma; garbage(); ltop = avma;
gel(z,2) = HUGE(); stackdummy(av, ltop);
```

Compared to the orthodox

```
gel(z,2) = gerepile(av, ltop, gel(z,2));
```

or even more wasteful

```
z = gerepilecopy(av0, z);
```

we temporarily lose $(av - ltop)$ words but save a costly `gerepile`. In principle, a garbage collection higher up the call chain should reclaim this later anyway.

Without the `stackdummy`, if the `[av, ltop]` zone is arbitrary (not even valid GENs as could happen after direct truncation via `setlg`), we would leave dangerous data in the middle of `z`, which would be a problem for a later

```
gerepile(..., ... , z);
```

And even if it were made of valid GENs, inhibiting the area makes sure `gerepile` will not inspect their components, saving time.

Another natural use in low-level routines is to “shorten” an existing GEN `z` to its first $n - 1$ components:

```
setlg(z, n);
stackdummy((pari_sp)(z + lg(z)), (pari_sp)(z + n));
```

or to its last n components:

```
long L = lg(z) - n, tz = typ(z);
stackdummy((pari_sp)(z + L), (pari_sp)z);
z += L; z[0] = evaltyp(tz) | evallg(L);
```

The first scenario (safe shortening an existing GEN) is in fact so common, that we provide a function for this:

`void fixlg(GEN z, long ly)` a safe variant of `setlg(z, ly)`. If `ly` is larger than `lg(z)` do nothing. Otherwise, shorten `z` in place, using `stackdummy` to avoid later `gerepile` problems.

`GEN gcopy_avma(GEN x, pari_sp *AVMA)` return a copy of x as from `gcopy`, except that we pretend that initially `avma` is `*AVMA`, and that `*AVMA` is updated accordingly (so that the total size of x is the difference between the two successive values of `*AVMA`). It is not necessary for `*AVMA` to initially point on the stack: `gclone` is implemented using this mechanism.

`GEN icopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a `t_INT` and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

5.5.5 Debugging the PARI stack.

`int chk_gerepileupto(GEN x)` returns 1 if x is suitable for `gerepileupto`, and 0 otherwise. In the latter case, print a warning explaining the problem.

`void dbg_gerepile(pari_sp ltop)` outputs the list of all objects on the stack between `avma` and `ltop`, i.e. the ones that would be inspected in a call to `gerepile(..., ltop, ...)`.

`void dbg_gerepileupto(GEN q)` outputs the list of all objects on the stack that would be inspected in a call to `gerepileupto(..., q)`.

5.5.6 Copies.

`GEN gcopy(GEN x)` creates a new copy of x on the stack.

`GEN gcopy_lg(GEN x, long l)` creates a new copy of x on the stack, pretending that `lg(x)` is `l`, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `gcopy(x)`.

`int isonstack(GEN x)` true iff x belongs to the stack.

`void copyifstack(GEN x, GEN y)` sets `y = gcopy(x)` if x belongs to the stack, and `y = x` otherwise. This macro evaluates its arguments once, contrary to

```
y = isonstack(x)? gcopy(x): x;
```

`void icopyifstack(GEN x, GEN y)` as `copyifstack` assuming x is a `t_INT`.

5.5.7 Simplify.

`GEN simplify(GEN x)` you should not need that function in library mode. One rather uses:

`GEN simplify_shallow(GEN x)` shallow, faster, version of `simplify`.

5.6 The PARI heap.

5.6.1 Introduction.

It is implemented as a doubly-linked list of `malloc`'ed blocks of memory, equipped with reference counts. Each block has type `GEN` but need not be a valid `GEN`: it is a chunk of data preceded by a hidden header (meaning that we allocate x and return $x + \text{headersize}$). A *clone*, created by `gclone`, is a block which is a valid `GEN` and whose *clone bit* is set.

5.6.2 Public interface.

`GEN newblock(size_t n)` allocates a block of n words (not bytes).

`void killblock(GEN x)` deletes the block x created by `newblock`. Fatal error if x not a block.

`GEN gclone(GEN x)` creates a new permanent copy of x on the heap (allocated using `newblock`). The *clone bit* of the result is set.

`GEN gcloneref(GEN x)` if x is not a clone, clone it and return the result; otherwise, increase the clone reference count and return x .

`void gunclone(GEN x)` deletes a clone. Deletion at first only decreases the reference count by 1. If the count remains positive, no further action is taken; if the count becomes zero, then the clone is actually deleted. In the current implementation, this is an alias for `killblock`, but it is cleaner to kill clones (valid `GENs`) using this function, and other blocks using `killblock`.

`void gunclone_deep(GEN x)` is only useful in the context of the GP interpreter which may replace arbitrary components of container types (`t_VEC`, `t_COL`, `t_MAT`, `t_LIST`) by clones. If x is such a container, the function recursively deletes all clones among the components of x , then unclones x . Useless in library mode: simply use `gunclone`.

`void traverseheap(void(*f)(GEN, void*), void *data)` this applies $f(x, \text{data})$ to each object x on the PARI heap, most recent first. Mostly for debugging purposes.

`GEN getheap()` a simple wrapper around `traverseheap`. Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words.

`GEN cgetg_block(long x, long y)` as `cgetg(x,y)`, creating the return value as a block, not on the PARI stack.

`GEN cgetr_block(long prec)` as `cgetr(prec)`, creating the return value as a block, not on the PARI stack.

5.6.3 Implementation note. The hidden block header is manipulated using the following private functions:

`void* bl_base(GEN x)` returns the pointer that was actually allocated by `malloc` (can be freed).

`long bl_refc(GEN x)` the reference count of x : the number of pointers to this block. Decrement in `killblock`, incremented by the private function `void gclone_refc(GEN x)`; block is freed when the reference count reaches 0.

`long bl_num(GEN x)` the index of this block in the list of all blocks allocated so far (including freed blocks). Uniquely identifies a block until $2^{\text{BITS_IN_LONG}}$ blocks have been allocated and this wraps around.

GEN `bl_next(GEN x)` the block *after* x in the linked list of blocks (NULL if x is the last block allocated not yet killed).

GEN `bl_prev(GEN x)` the block allocated *before* x (never NULL).

We documented the last four routines as functions for clarity (and type checking) but they are actually macros yielding valid lvalues. It is allowed to write `bl_refc(x)++` for instance.

5.7 Handling user and temp variables.

Low-level implementation of user / temporary variables is liable to change. We describe it nevertheless for completeness. Currently variables are implemented by a single array of values divided in 3 zones: `0–nvar` (user variables), `max_avail–MAXVARN` (temporary variables), and `nvar+1–max_avail-1` (pool of free variable numbers).

5.7.1 Low-level.

`void pari_var_init()`: a small part of `pari_init`. Resets variable counters `nvar` and `max_avail`, notwithstanding existing variables! In effect, this even deletes x . Don't use it.

`void pari_var_close(void)` attached destructor, called by `pari_close`.

`long pari_var_next()`: returns `nvar`, the number of the next user variable we can create.

`long pari_var_next_temp()` returns `max_avail`, the number of the next temp variable we can create.

`long pari_var_create(entree *ep)` low-level initialization of an `EpVAR`. Return the attached (new) variable number.

GEN `vars_sort_inplace(GEN z)` given a `t_VECSMALL` z of variable numbers, sort z in place according to variable priorities (highest priority comes first).

GEN `vars_to_RgXV(GEN h)` given a `t_VECSMALL` z of variable numbers, return the `t_VEC` of `pol_x(z[i])`.

5.7.2 User variables.

`long fetch_user_var(char *s)` returns a user variable whose name is s , creating it is needed (and using an existing variable otherwise). Returns its variable number.

GEN `fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered v . Return NULL for a temporary variable.

`entree* is_entry(const char *s)` returns the `entree*` attached to an identifier s (variable or function), from the interpreter hashtables. Return NULL is the identifier is unknown.

5.7.3 Temporary variables.

`long fetch_var(void)` returns the number of a new temporary variable (decreasing `max_avail`).

`long delete_var(void)` delete latest temp variable created and return the number of previous one.

`void name_var(long n, char *s)` rename temporary variable number n to s ; mostly useful for nicer printout. Error when trying to rename a user variable.

5.8 Adding functions to PARI.

5.8.1 Nota Bene. As mentioned in the `COPYING` file, modified versions of the PARI package can be distributed under the conditions of the GNU General Public License. If you do modify PARI, however, it is certainly for a good reason, and we would like to know about it, so that everyone can benefit from your changes. There is then a good chance that your improvements are incorporated into the next release.

We classify changes to PARI into four rough classes, where changes of the first three types are almost certain to be accepted. The first type includes all improvements to the documentation, in a broad sense. This includes correcting typos or inaccuracies of course, but also items which are not really covered in this document, e.g. if you happen to write a tutorial, or pieces of code exemplifying fine points unduly omitted in the present manual.

The second type is to expand or modify the configuration routines and skeleton files (the `Configure` script and anything in the `config/` subdirectory) so that compilation is possible (or easier, or more efficient) on an operating system previously not catered for. This includes discovering and removing idiosyncrasies in the code that would hinder its portability.

The third type is to modify existing (mathematical) code, either to correct bugs, to add new functionality to existing functions, or to improve their efficiency.

Finally the last type is to add new functions to PARI. We explain here how to do this, so that in particular the new function can be called from `gp`.

5.8.2 Coding guidelines. Code your function in a file of its own, using as a guide other functions in the PARI sources. One important thing to remember is to clean the stack before exiting your main function, since otherwise successive calls to the function clutters the stack with unnecessary garbage, and stack overflow occurs sooner. Also, if it returns a `GEN` and you want it to be accessible to `gp`, you have to make sure this `GEN` is suitable for `gerepileupto` (see Section 4.3).

If error messages or warnings are to be generated in your function, use `pari_err` and `pari_warn` respectively. Recall that `pari_err` does not return but ends with a `longjmp` statement. As well, instead of explicit `printf` / `fprintf` statements, use the following encapsulated variants:

`void pari_putc(char c):` write character `c` to the output stream.

`void pari_puts(char *s):` write `s` to the output stream.

`void pari_printf(const char *fmt, ...):` write following arguments to the output stream, according to the conversion specifications in format `fmt` (see `printf`).

`void err_printf(const char *fmt, ...):` as `pari_printf`, writing to PARI's current error stream.

`void err_flush(void)` flush error stream.

Declare all public functions in an appropriate header file, if you want to access them from `C`. The other functions should be declared `static` in your file.

Your function is now ready to be used in library mode after compilation and creation of the library. If possible, compile it as a shared library (see the `Makefile` coming with the `extgcd` example in the distribution). It is however still inaccessible from `gp`.

5.8.3 GP prototypes, parser codes. A *GP prototype* is a character string describing all the GP parser needs to know about the function prototype. It contains a sequence of the following atoms:

- Return type: `GEN` by default (must be valid for `gerepileupto`), otherwise the following can appear as the *first* char of the code string:

```

i      return int
l      return long
u      return ulong
v      return void
m      return a GEN which is not gerepile-safe.
```

The `m` code is used for member functions, to avoid unnecessary copies. A copy opcode is generated by the compiler if the result needs to be kept safe for later use.

- Mandatory arguments, appearing in the same order as the input arguments they describe:

```

G      GEN
&      *GEN
L      long (we implicitly typecast int to long)
U      ulong
V      loop variable
n      variable, expects a variable number (a long, not an *entree)
W      a GEN which is a lvalue to be modified in place (for t_LIST)
r      raw input (treated as a string without quotes). Quoted args are copied as strings
        Stops at first unquoted ')' or ','. Special chars can be quoted using '\'.
        Example: aa"b\n)"c yields the string "aab\n)c"
s      expanded string. Example: Pi"x"2 yields "3.142x2"
        Unquoted components can be of any PARI type, converted to string following
        current output format
I      closure whose value is ignored, as in for loops,
        to be processed by void closure_evalvoid(GEN C)
E      closure whose value is used, as in sum loops,
        to be processed by void closure_evalgen(GEN C)
J      implicit function of arity 1, as in parsum loops,
        to be processed by void closure_callgen1(GEN C)
```

A *closure* is a GP function in compiled (bytecode) form. It can be efficiently evaluated using the `closure_evalxxx` functions.

- Automatic arguments:

```

f      Fake *long. C function requires a pointer but we do not use the resulting long
b      current real precision in bits
p      current real precision in words
P      series precision (default seriesprecision, global variable precdl for the library)
C      lexical context (internal, for eval, see localvars_read_str)
```

- Syntax requirements, used by functions like `for`, `sum`, etc.:
 - = separator = required at this point (between two arguments)

- Optional arguments and default values:

```

E*     any number of expressions, possibly 0 (see E)
s*     any number of strings, possibly 0 (see s)
```

Dxxx argument can be omitted and has a default value

The **E*** code reads all remaining arguments in closure context and passes them as a single `t_VEC`. The **s*** code reads all remaining arguments in *string context* and passes the list of strings as a single `t_VEC`. The automatic concatenation rules in string context are implemented so that adjacent strings are read as different arguments, as if they had been comma-separated. For instance, if the remaining argument sequence is: "xx" 1, "yy", the **s*** atom sends [a, b, c], where *a*, *b*, *c* are GENs of type `t_STR` (content "xx"), `t_INT` (equal to 1) and `t_STR` (content "yy").

The format to indicate a default value (atom starts with a D) is "D*value, type,*", where *type* is the code for any mandatory atom (previous group), *value* is any valid GP expression which is converted according to *type*, and the ending comma is mandatory. For instance `D0,L`, stands for "this optional argument is converted to a long, and is 0 by default". So if the user-given argument reads 1 + 3 at this point, `4L` is sent to the function; and `0L` if the argument is omitted. The following special notations are available:

- DG optional GEN, send NULL if argument omitted.
- D& optional *GEN, send NULL if argument omitted.
The argument must be prefixed by &.
- DI, DE optional closure, send NULL if argument omitted.
- DP optional long, send `precd1` if argument omitted.
- DV optional *entree, send NULL if argument omitted.
- Dn optional variable number, -1 if omitted.
- Dr optional raw string, send NULL if argument omitted.
- Ds optional char *, send NULL if argument omitted.

Hardcoded limit. C functions using more than 20 arguments are not supported. Use vectors if you really need that many parameters.

When the function is called under `gp`, the prototype is scanned and each time an atom corresponding to a mandatory argument is met, a user-given argument is read (`gp` outputs an error message if the argument was missing). Each time an optional atom is met, a default value is inserted if the user omits the argument. The "automatic" atoms fill in the argument list transparently, supplying the current value of the corresponding variable (or a dummy pointer).

For instance, here is how you would code the following prototypes, which do not involve default values:

```
GEN f(GEN x, GEN y, long prec)  ----> "GGp"
void f(GEN x, GEN y, long prec) ----> "vGGp"
void f(GEN x, long y, long prec) ----> "vGLp"
long f(GEN x)                   ----> "lG"
int f(long x)                   ----> "iL"
```

If you want more examples, `gp` gives you easy access to the parser codes attached to all GP functions: just type `\h function`. You can then compare with the C prototypes as they stand in `paridecl.h`.

Remark. If you need to implement complicated control statements (probably for some improved summation functions), you need to know how the parser implements closures and lexicals and how the evaluator lets you deal with them, in particular the `push_lex` and `pop_lex` functions. Check their descriptions and adapt the source code in `language/sumiter.c` and `language/intnum.c`.

5.8.4 Integration with `gp` as a shared module.

In this section we assume that your Operating System is supported by `install`. You have written a function in C following the guidelines in Section 5.8.2; in case the function returns a `GEN`, it must satisfy `gerepileupto` assumptions (see Section 4.3).

You then succeeded in building it as part of a shared library and want to finally tell `gp` about your function. First, find a name for it. It does not have to match the one used in library mode, but consistency is nice. It has to be a valid GP identifier, i.e. use only alphabetic characters, digits and the underscore character (`_`), the first character being alphabetic.

Then figure out the correct parser code corresponding to the function prototype (as explained in Section 5.8.3) and write a GP script like the following:

```
install(libname, code, gpname, library)
addhelp(gpname, "some help text")
```

The `addhelp` part is not mandatory, but very useful if you want others to use your module. `libname` is how the function is named in the library, usually the same name as one visible from C.

Read that file from your `gp` session, for instance from your preferences file (or `gprc`), and that's it. You can now use the new function `gpname` under `gp`, and we would very much like to hear about it!

Example. A complete description could look like this:

```
{
  install(bnfinit0, "GD0,L,DGp", ClassGroupInit, "libpari.so");
  addhelp(ClassGroupInit, "ClassGroupInit(P,{flag=0},{data=[]}):
    compute the necessary data for ...");
}
```

which means we have a function `ClassGroupInit` under `gp`, which calls the library function `bnfinit0`. The function has one mandatory argument, and possibly two more (two 'D' in the code), plus the current real precision. More precisely, the first argument is a `GEN`, the second one is converted to a `long` using `itos` (0 is passed if it is omitted), and the third one is also a `GEN`, but we pass `NULL` if no argument was supplied by the user. This matches the C prototype (from `paridecl.h`):

```
GEN bnfinit0(GEN P, long flag, GEN data, long prec)
```

This function is in fact coded in `basemath/buch2.c`, and is in this case completely identical to the GP function `bnfinit` but `gp` does not need to know about this, only that it can be found somewhere in the shared library `libpari.so`.

Important note. You see in this example that it is the function's responsibility to correctly interpret its operands: `data = NULL` is interpreted *by the function* as an empty vector. Note that since `NULL` is never a valid `GEN` pointer, this trick always enables you to distinguish between a default value and actual input: the user could explicitly supply an empty vector!

5.8.5 Library interface for `install`.

There is a corresponding library interface for this `install` functionality, letting you expand the GP parser/evaluator available in the library with new functions from your C source code. Functions such as `gp_read_str` may then evaluate a GP expression sequence involving calls to these new function!

```
entree * install(void *f, const char *gpname, const char *code)
```

where `f` is the (address of the) function (cast to `void*`), `gpname` is the name by which you want to access your function from within your GP expressions, and `code` is as above.

5.8.6 Integration by patching `gp`.

If `install` is not available, and installing Linux or a BSD operating system is not an option (why?), you have to hardcode your function in the `gp` binary. Here is what needs to be done:

- Fetch the complete sources of the PARI distribution.
- Drop the function source code module in an appropriate directory (a priori `src/modules`), and declare all public functions in `src/headers/paridecl.h`.
- Choose a help section and add a file `src/functions/section/gpname` containing the following, keeping the notation above:

```
Function:  gpname
Section:   section
C-Name:    libname
Prototype: code
Help:      some help text
```

(If the help text does not fit on a single line, continuation lines must start by a whitespace character.) Two GP2C-related fields (`Description` and `Wrapper`) are also available to improve the code GP2C generates when compiling scripts involving your function. See the GP2C documentation for details.

- Launch `Configure`, which should pick up your C files and build an appropriate `Makefile`. At this point you can recompile `gp`, which will first rebuild the functions database.

Example. We reuse the `ClassGroupInit` / `bnfinit0` from the preceding section. Since the C source code is already part of PARI, we only need to add a file

```
functions/number_fields/ClassGroupInit
```

containing the following:

```
Function: ClassGroupInit
Section:  number_fields
C-Name:   bnfinit0
Prototype: GD0,L,DGp
Help:    ClassGroupInit(P,{flag=0},{tech=[]}): this routine does ...
```

and recompile `gp`.

5.9 Globals related to PARI configuration.

5.9.1 PARI version numbers.

`paricfg_version_code` encodes in a single `long`, the Major and minor version numbers as well as the patchlevel.

`long PARI_VERSION(long M, long m, long p)` produces the version code attached to release $M.m.p$. Each code identifies a unique PARI release, and corresponds to the natural total order on the set of releases (bigger code number means more recent release).

`PARI_VERSION_SHIFT` is the number of bits used to store each of the integers M, m, p in the version code.

`paricfg_vcversion` is a version string related to the revision control system used to handle your sources, if any. For instance `git-commit hash` if compiled from a git repository.

The two character strings `paricfg_version` and `paricfg_buildinfo`, correspond to the first two lines printed by `gp` just before the Copyright message. The character string `paricfg_compiledate` is the date of compilation which appears on the next line. The character string `paricfg_mt_engine` is the name of the threading engine on the next line.

`GEN pari_version()` returns the version number as a PARI object, a `t_VEC` with three `t_INT` and one `t_STR` components.

5.9.2 Miscellaneous.

`paricfg_datadir`: character string. The location of PARI's `datadir`.

`paricfg_gphelp`: character string. The name of an external help command for ?? (such as the `gphelp` script)

Chapter 6: Arithmetic kernel: Level 0 and 1

6.1 Level 0 kernel (operations on ulongs).

6.1.1 Micro-kernel. The Level 0 kernel simulates basic operations of the 68020 processor on which PARI was originally implemented. They need “global” `ulong` variables `overflow` (which will contain only 0 or 1) and `hiremainder` to function properly. A routine using one of these lowest-level functions where the description mentions either `hiremainder` or `overflow` must declare the corresponding

```
LOCAL_HIREMAINDER; /* provides 'hiremainder' */
LOCAL_OVERFLOW;    /* provides 'overflow' */
```

in a declaration block. Variables `hiremainder` and `overflow` then become available in the enclosing block. For instance a loop over the powers of an `ulong` `p` protected from overflows could read

```
while (pk < lim)
{
    LOCAL_HIREMAINDER;
    ...
    pk = mulll(pk, p); if (hiremainder) break;
}
```

For most architectures, the functions mentioned below are really chunks of inlined assembler code, and the above ‘global’ variables are actually local register values.

`ulong addll(ulong x, ulong y)` adds `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong addllx(ulong x, ulong y)` adds `overflow` to the sum of the `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong subll(ulong x, ulong y)` subtracts `x` and `y`, returns the lower `BITS_IN_LONG` bits and put the carry (borrow) bit into `overflow`.

`ulong subllx(ulong x, ulong y)` subtracts `overflow` from the difference of `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry (borrow) bit into `overflow`.

`int bfffo(ulong x)` returns the number of leading zero bits in `x`. That is, the number of bit positions by which it would have to be shifted left until its leftmost bit first becomes equal to 1, which can be between 0 and `BITS_IN_LONG - 1` for nonzero `x`. When `x` is 0, the result is undefined.

`ulong mulll(ulong x, ulong y)` multiplies `x` by `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong addmul(ulong x, ulong y)` adds `hiremainder` to the product of `x` and `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong divll(ulong x, ulong y)` returns the quotient of $(\text{hiremainder} * 2^{\text{BITS_IN_LONG}}) + x$ by y and stores the remainder into `hiremainder`. An error occurs if the quotient cannot be represented by an `ulong`, i.e. if initially $\text{hiremainder} \geq y$.

`long hammingl(ulong x)` returns the Hamming weight of x , i.e. the number of non-zero bits in its binary expansion.

Obsolete routines. Those functions are awkward and no longer used; they are only provided for backward compatibility:

`ulong shiftl(ulong x, ulong y)` returns x shifted left by y bits, i.e. $x \ll y$, where we assume that $0 \leq y \leq \text{BITS_IN_LONG}$. The global variable `hiremainder` receives the bits that were shifted out, i.e. $x \gg (\text{BITS_IN_LONG} - y)$.

`ulong shiftr(ulong x, ulong y)` returns x shifted right by y bits, i.e. $x \gg y$, where we assume that $0 \leq y \leq \text{BITS_IN_LONG}$. The global variable `hiremainder` receives the bits that were shifted out, i.e. $x \ll (\text{BITS_IN_LONG} - y)$.

6.1.2 Modular kernel. The following routines are not part of the level 0 kernel per se, but implement modular operations on words in terms of the above. They are written so that no overflow may occur. Let $m \geq 1$ be the modulus; all operands representing classes modulo m are assumed to belong to $[0, m - 1]$. The result may be wrong for a number of reasons otherwise: it may not be reduced, overflow can occur, etc.

`int odd(ulong x)` returns 1 if x is odd, and 0 otherwise.

`int both_odd(ulong x, ulong y)` returns 1 if x and y are both odd, and 0 otherwise.

`ulong invmod2BIL(ulong x)` returns the smallest positive representative of $x^{-1} \bmod 2^{\text{BITS_IN_LONG}}$, assuming x is odd.

`ulong Fl_add(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of $x + y$ modulo m .

`ulong Fl_neg(ulong x, ulong m)` returns the smallest non-negative representative of $-x$ modulo m .

`ulong Fl_sub(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of $x - y$ modulo m .

`long Fl_center(ulong x, ulong m, ulong mo2)` returns the representative in $] -m/2, m/2]$ of x modulo m . Assume $0 \leq x < m$ and $\text{mo2} = m \gg 1$.

`ulong Fl_mul(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of xy modulo m .

`ulong Fl_double(ulong x, ulong m)` returns $2x$ modulo m .

`ulong Fl_triple(ulong x, ulong m)` returns $3x$ modulo m .

`ulong Fl_half(ulong x, ulong m)` returns z such that $2z = x$ modulo m assuming such z exists.

`ulong Fl_sqr(ulong x, ulong m)` returns the smallest non-negative representative of x^2 modulo m .

`ulong Fl_inv(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m . If x is not invertible mod m , raise an exception.

`ulong Fl_invsafe(ulong x, ulong m)` returns the smallest positive representative of x^{-1} modulo m . If x is not invertible mod m , return 0 (which is ambiguous if $m = 1$).

`ulong Fl_invgen(ulong x, ulong m, ulong *pg)` set `*pg` to $g = \gcd(x, m)$ and return u in $(\mathbf{Z}/m\mathbf{Z})^*$ such that $xu = g$ modulo m . We have $g = 1$ if and only if x is invertible, and in this case u is its inverse.

`ulong Fl_div(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of xy^{-1} modulo m . If y is not invertible mod m , raise an exception.

`ulong Fl_powu(ulong x, ulong n, ulong m)` returns the smallest non-negative representative of x^n modulo m .

`GEN Fl_powers(ulong x, long n, ulong p)` returns $[x^0, \dots, x^n]$ modulo m , as a `t_VECSMALL`.

`ulong Fl_sqrt(ulong x, ulong p)` returns the square root of x modulo p (smallest non-negative representative). Assumes p to be prime, and x to be a square modulo p .

`ulong Fl_sqrtl(ulong x, ulong l, ulong p)` returns a l -th root of x modulo p . Assumes p to be prime and $p \equiv 1 \pmod{l}$, and x to be a l -th power modulo p .

`ulong Fl_sqrtn(ulong a, ulong n, ulong p, ulong *zn)` returns `ULONG_MAX` if a is not an n -th power residue mod p . Otherwise, returns an n -th root of a ; if `zn` is non-NULL set it to a primitive m -th root of 1, $m = \gcd(p-1, n)$ allowing to compute all m solutions in \mathbf{F}_p of the equation $x^n = a$.

`ulong Fl_log(ulong a, ulong g, ulong ord, ulong p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is undefined.

`ulong Fl_order(ulong a, ulong o, ulong p)` returns the order of the \mathbf{F}_p a . It is assumed that o is a multiple of the order of a , 0 being allowed (no non-trivial information).

`ulong random_Fl(ulong p)` returns a pseudo-random integer uniformly distributed in $0, 1, \dots, p-1$.

`ulong pgener_Fl(ulong p)` returns the smallest primitive root modulo p , assuming p is prime.

`ulong pgener_Zl(ulong p)` returns the smallest primitive root modulo p^k , $k > 1$, assuming p is an odd prime.

`ulong pgener_Fl_local(ulong p, GEN L)`, see `gener_Fp_local`, L is an `Flv`.

6.1.3 Modular kernel with “precomputed inverse”.

This is based on an algorithm by T. Grandlund and N. Möller in “Improved division by invariant integers” <http://gmplib.org/~tege/division-paper.pdf>.

In the following, we set $B = \text{BITS_IN_LONG}$.

`ulong get_Fl_red(ulong p)` returns a pseudo inverse pi for p

`ulong divll_pre(ulong x, ulong p, ulong yi)` as `divll`, where yi is the pseudo inverse of y .

`ulong remll_pre(ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of $u_1 2^B + u_0$ modulo p , assuming pi is the pseudo inverse of p . This function is faster if $u_1 < p$.

`ulong remlll_pre(ulong u2, ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of $u_2 2^{2B} + u_1 2^B + u_0$ modulo p , assuming pi is the pseudo inverse of p .

`ulong Fl_sqr_pre(ulong x, ulong p, ulong pi)` returns x^2 modulo p , assuming pi is the pseudo inverse of p .

`ulong Fl_mul_pre(ulong x, ulong y, ulong p, ulong pi)` returns xy modulo p , assuming pi is the pseudo inverse of p .

`ulong Fl_addmul_pre(ulong a, ulong b, ulong c, ulong p, ulong pi)` returns $a+bc$ modulo p , assuming pi is the pseudo inverse of p .

`ulong Fl_addmulmul_pre(ulong a, ulong b, ulong c, ulong d, ulong p, ulong pi)` returns $ab+cd$ modulo p , assuming pi is the pseudo inverse of p .

`ulong Fl_powu_pre(ulong x, ulong n, ulong p, ulong pi)` returns x^n modulo p , assuming pi is the pseudo inverse of p .

`GEN Fl_powers_pre(ulong x, long n, ulong p, ulong pi)` returns the vector (`t_VECSMALL`) (x^0, \dots, x^n) , assuming pi is the pseudo inverse of p .

`ulong Fl_log_pre(ulong a, ulong g, ulong ord, ulong p, ulong pi)` as `Fl_log`, assuming pi is the pseudo inverse of p .

`ulong Fl_sqrt_pre(ulong x, ulong p, ulong pi)` returns a square root of x modulo p , assuming pi is the pseudo inverse of p . See `Fl_sqrt`.

`ulong Fl_sqrtl_pre(ulong x, ulong l, ulong p, ulong pi)` returns a l -th root of x modulo p , assuming pi is the pseudo inverse of p , p prime and $p \equiv 1 \pmod{l}$, and x to be a l -th power modulo p .

`ulong Fl_sqrtn_pre(ulong x, ulong n, ulong p, ulong *zn)` See `Fl_sqrtn`, assuming pi is the pseudo inverse of p .

`ulong Fl_2gener_pre(ulong p, ulong pi)` return a generator of the 2-Sylow subgroup of \mathbf{F}_p^* . To use with `Fl_sqrt_pre_i`.

`ulong Fl_sqrt_pre_i(ulong x, ulong s2, ulong p, ulong pi)` as `Fl_sqrt_pre` where $s2$ is the element returned by `Fl_2gener_pre`.

6.1.4 Switching between `Fl_xxx` and standard operators.

Even though the `Fl_xxx` routines are efficient, they are slower than ordinary long operations, using the standard `+`, `%`, etc. operators. The following macro is used to choose in a portable way the most efficient functions for given operands:

`int SMALL_ULONG(ulong p)` true if $2p^2 < 2^{\text{BITS_IN_LONG}}$. In that case, it is possible to use ordinary operators efficiently. If $p < 2^{\text{BITS_IN_LONG}}$, one may still use the `Fl_xxx` routines. Otherwise, one must use generic routines. For instance, the scalar product of the GENS x and y mod p could be computed as follows.

```

long i, l = lg(x);
if (lgefint(p) > 3)
{ /* arbitrary */
  GEN s = gen_0;
  for (i = 1; i < l; i++) s = addii(s, mulii(gel(x,i), gel(y,i)));
  return modii(s, p).
}
else
{
  ulong s = 0, pp =itou(p);

```

```

x = ZV_to_Flv(x, pp);
y = ZV_to_Flv(y, pp);
if (SMALL_ULONG(pp))
{ /* very small */
  for (i = 1; i < l; i++)
  {
    s += x[i] * y[i];
    if (s & HIGHBIT) s %= pp;
  }
  s %= pp;
}
else
{ /* small */
  for (i = 1; i < l; i++)
    s = Fl_add(s, Fl_mul(x[i], y[i], pp), pp);
}
return utoi(s);
}

```

In effect, we have three versions of the same code: very small, small, and arbitrary inputs. The very small and arbitrary variants use lazy reduction and reduce only when it becomes necessary: when overflow might occur (very small), and at the very end (very small, arbitrary).

6.2 Level 1 kernel (operations on longs, integers and reals).

Note. Some functions consist of an elementary operation, immediately followed by an assignment statement. They will be introduced as in the following example:

`GEN gadd[z](GEN x, GEN y[, GEN z])` followed by the explicit description of the function

`GEN gadd(GEN x, GEN y)`

which creates its result on the stack, returning a GEN pointer to it, and the parts in brackets indicate that there exists also a function

`void gaddz(GEN x, GEN y, GEN z)`

which assigns its result to the pre-existing object `z`, leaving the stack unchanged. These assignment variants are kept for backward compatibility but are inefficient: don't use them.

6.2.1 Creation.

`GEN cgeti(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetipos(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to 1.

`GEN cgetineg(long n)` allocates memory on the PARI stack for a negative `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to `-1`.

`GEN cgetr(long n)` allocates memory on the PARI stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_REAL)`.

GEN `cgetc(long n)` allocates memory on the PARI stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REALs` of length `n`.

GEN `real_1(long prec)` create a `t_REAL` equal to 1 to `prec` words of accuracy.

GEN `real_1_bit(long bitprec)` create a `t_REAL` equal to 1 to `bitprec` bits of accuracy.

GEN `real_m1(long prec)` create a `t_REAL` equal to -1 to `prec` words of accuracy.

GEN `real_0_bit(long bit)` create a `t_REAL` equal to 0 with exponent $-\text{bit}$.

GEN `real_0(long prec)` is a shorthand for

`real_0_bit(-prec2nbits(prec))`

GEN `int2n(long n)` creates a `t_INT` equal to $1 \ll n$ (i.e 2^n if $n \geq 0$, and 0 otherwise).

GEN `int2u(ulong n)` creates a `t_INT` equal to 2^n .

GEN `int2um1(long n)` creates a `t_INT` equal to $2^n - 1$.

GEN `real2n(long n, long prec)` create a `t_REAL` equal to 2^n to `prec` words of accuracy.

GEN `real_m2n(long n, long prec)` create a `t_REAL` equal to -2^n to `prec` words of accuracy.

GEN `strtoi(char *s)` convert the character string `s` to a non-negative `t_INT`. Decimal numbers, hexadecimal numbers prefixed by `0x` and binary numbers prefixed by `0b` are allowed. The string `s` consists exclusively of digits: no leading sign, no whitespace. Leading zeroes are discarded.

GEN `strtor(char *s, long prec)` convert the character string `s` to a non-negative `t_REAL` of precision `prec`. The string `s` consists exclusively of digits and optional decimal point and exponent (`e` or `E`): no leading sign, no whitespace. Leading zeroes are discarded.

6.2.2 Assignment. In this section, the `z` argument in the `z`-functions must be of type `t_INT` or `t_REAL`.

`void mpaff(GEN x, GEN z)` assigns `x` into `z` (where `x` and `z` are `t_INT` or `t_REAL`). Assumes that $\text{lg}(z) > 2$.

`void affii(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_INT` `z`.

`void affir(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_REAL` `z`. Assumes that $\text{lg}(z) > 2$.

`void affiz(GEN x, GEN z)` assigns `t_INT` `x` into `t_INT` or `t_REAL` `z`. Assumes that $\text{lg}(z) > 2$.

`void affsi(long s, GEN z)` assigns the `long` `s` into the `t_INT` `z`. Assumes that $\text{lg}(z) > 2$.

`void affsr(long s, GEN z)` assigns the `long` `s` into the `t_REAL` `z`. Assumes that $\text{lg}(z) > 2$.

`void affsz(long s, GEN z)` assigns the `long` `s` into the `t_INT` or `t_REAL` `z`. Assumes that $\text{lg}(z) > 2$.

`void affui(ulong u, GEN z)` assigns the `ulong` `u` into the `t_INT` `z`. Assumes that $\text{lg}(z) > 2$.

`void affur(ulong u, GEN z)` assigns the `ulong` `u` into the `t_REAL` `z`. Assumes that $\text{lg}(z) > 2$.

`void affrr(GEN x, GEN z)` assigns the `t_REAL` `x` into the `t_REAL` `z`.

`void affgr(GEN x, GEN z)` assigns the scalar `x` into the `t_REAL` `z`, if possible.

The function `affrs` and `affri` do not exist. So don't use them.

`void affrr_fixlg(GEN y, GEN z)` a variant of `affrr`. First shorten z so that it is no longer than y , then assigns y to z . This is used in the following scenario: room is reserved for the result but, due to cancellation, fewer words of accuracy are available than had been anticipated; instead of appending meaningless 0s to the mantissa, we store what was actually computed.

Note that shortening z is not quite straightforward, since `setlg(z, ly)` would leave garbage on the stack, which `gerepile` might later inspect. It is done using

`void fixlg(GEN z, long ly)` see `stackdummy` and the examples that follow.

6.2.3 Copy.

`GEN icopy(GEN x)` copy relevant words of the `t_INT` x on the stack: the length and effective length of the copy are equal.

`GEN rcopy(GEN x)` copy the `t_REAL` x on the stack.

`GEN leafcopy(GEN x)` copy the leaf x on the stack (works in particular for `t_INTs` and `t_REALs`). Contrary to `icopy`, `leafcopy` preserves the original length of a `t_INT`. The obsolete form `GEN mpcopy(GEN x)` is still provided for backward compatibility.

This function also works on recursive types, copying them as if they were leaves, i.e. making a shallow copy in that case: the components of the copy point to the same data as the component of the source; see also `shallowcopy`.

`GEN leafcopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume x is a leaf and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

`GEN icopyspec(GEN x, long nx)` copy the nx words $x[2], \dots, x[nx+1]$ to make up a new `t_INT`. Set the sign to 1.

6.2.4 Conversions.

`GEN itor(GEN x, long prec)` converts the `t_INT` x to a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

`long itos(GEN x)` converts the `t_INT` x to a `long` if possible, otherwise raise an exception. We consider the conversion to be possible if and only if $|x| \leq \text{LONG_MAX}$, i.e. $|x| < 2^{63}$ on a 64-bit architecture. Since the range is symmetric, the output of `itos` can safely be negated.

`long itos_or_0(GEN x)` converts the `t_INT` x to a `long` if possible, otherwise return 0.

`int is_bigint(GEN n)` true if `itos(n)` would give an error.

`ulong itou(GEN x)` converts the `t_INT` $|x|$ to an `ulong` if possible, otherwise raise an exception. The conversion is possible if and only if `lgefint(x) ≤ 3`.

`long itou_or_0(GEN x)` converts the `t_INT` $|x|$ to an `ulong` if possible, otherwise return 0.

`GEN stoi(long s)` creates the `t_INT` corresponding to the `long` s .

`GEN stor(long s, long prec)` converts the `long` s into a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

`GEN utoi(ulong s)` converts the `ulong` s into a `t_INT` and return the latter.

GEN `utoipos(ulong s)` converts the *non-zero* `ulong s` into a `t_INT` and return the latter.

GEN `utoineg(ulong s)` converts the *non-zero* `ulong s` into the `t_INT -s` and return the latter.

GEN `utor(ulong s, long prec)` converts the `ulong s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

GEN `rtor(GEN x, long prec)` converts the `t_REAL x` to a `t_REAL` of length `prec` and return the latter. If `prec < lg(x)`, round properly. If `prec > lg(x)`, pad with zeroes. Assumes that `prec > 2`.

The following function is also available as a special case of `mkintn`:

GEN `uu32toi(ulong a, ulong b)` returns the GEN equal to $2^{32}a + b$, *assuming* that $a, b < 2^{32}$. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

GEN `uu32toineg(ulong a, ulong b)` returns the GEN equal to $-(2^{32}a+b)$, *assuming* that $a, b < 2^{32}$ and that one of a or b is positive. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

GEN `uutoi(ulong a, ulong b)` returns the GEN equal to $2^{\text{BITS_IN_LONG}}a + b$.

GEN `uutoineg(ulong a, ulong b)` returns the GEN equal to $-(2^{\text{BITS_IN_LONG}}a + b)$.

6.2.5 Integer parts. The following four functions implement the conversion from `t_REAL` to `t_INT` using standard rounding modes. Contrary to usual semantics (complement the mantissa with an infinite number of 0), they will raise an error *precision loss in truncation* if the `t_REAL` represents a range containing more than one integer.

GEN `ceilr(GEN x)` smallest integer larger or equal to the `t_REAL x` (i.e. the `ceil` function).

GEN `floorr(GEN x)` largest integer smaller or equal to the `t_REAL x` (i.e. the `floor` function).

GEN `roundr(GEN x)` rounds the `t_REAL x` to the nearest integer (towards $+\infty$ in case of tie).

GEN `truncr(GEN x)` truncates the `t_REAL x` (not the same as `floorr` if x is negative).

The following four function are analogous, but can also treat the trivial case when the argument is a `t_INT`:

GEN `mpceil(GEN x)` as `ceilr` except that x may be a `t_INT`.

GEN `mpfloor(GEN x)` as `floorr` except that x may be a `t_INT`.

GEN `mpround(GEN x)` as `roundr` except that x may be a `t_INT`.

GEN `mptrunc(GEN x)` as `truncr` except that x may be a `t_INT`.

GEN `diviiround(GEN x, GEN y)` if x and y are `t_INTs`, returns the quotient x/y of x and y , rounded to the nearest integer. If x/y falls exactly halfway between two consecutive integers, then it is rounded towards $+\infty$ (as for `roundr`).

GEN `ceil_safe(GEN x)`, x being a real number (not necessarily a `t_REAL`) returns the smallest integer which is larger than any possible incarnation of x . (Recall that a `t_REAL` represents an interval of possible values.) Note that `gceil` raises an exception if the input accuracy is too low compared to its magnitude.

GEN `floor_safe(GEN x)`, x being a real number (not necessarily a `t_REAL`) returns the largest integer which is smaller than any possible incarnation of x . (Recall that a `t_REAL` represents an

interval of possible values.) Note that `gfloor` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN trunc_safe(GEN x)`, `x` being a real number (not necessarily a `t_REAL`) returns the integer with the largest absolute value, which is closer to 0 than any possible incarnation of `x`. (Recall that a `t_REAL` represents an interval of possible values.)

`GEN roundr_safe(GEN x)` rounds the `t_REAL` `x` to the nearest integer (towards $+\infty$). Complement the mantissa with an infinite number of 0 before rounding, hence never raise an exception.

6.2.6 2-adic valuations and shifts.

`long vals(long s)` 2-adic valuation of the `long` `s`. Returns -1 if `s` is equal to 0.

`long vali(GEN x)` 2-adic valuation of the `t_INT` `x`. Returns -1 if `x` is equal to 0.

`GEN mpshift(GEN x, long n)` shifts the `t_INT` or `t_REAL` `x` by `n`. If `n` is positive, this is a left shift, i.e. multiplication by 2^n . If `n` is negative, it is a right shift by $-n$, which amounts to the truncation of the quotient of `x` by 2^{-n} .

`GEN shifti(GEN x, long n)` shifts the `t_INT` `x` by `n`.

`GEN shiftr(GEN x, long n)` shifts the `t_REAL` `x` by `n`.

`void shiftr_inplace(GEN x, long n)` shifts the `t_REAL` `x` by `n`, in place.

`GEN trunc2nr(GEN x, long n)` given a `t_REAL` `x`, returns `truncr(shiftr(x,n))`, but faster, without leaving garbage on the stack and never raising a *precision loss in truncation* error. Called by `gtrunc2n`.

`GEN trunc2nr_lg(GEN x, long lx, long n)` given a `t_REAL` `x`, returns `trunc2nr(x,n)`, pretending that the length of `x` is `lx`, which must be $\leq \lg(x)$.

`GEN mantissa2nr(GEN x, long n)` given a `t_REAL` `x`, returns the mantissa of $x2^n$ (disregards the exponent of `x`). Equivalent to

`trunc2nr(x, n-expo(x)+bit_prec(x)-1)`

`GEN mantissa_real(GEN z, long *e)` returns the mantissa `m` of `z`, and sets `*e` to the exponent `bit_accuracy(lg(z)) - 1 - expo(z)`, so that $z = m/2^e$.

Low-level. In the following two functions, `s`(ource) and `t`(arget) need not be valid `GENs` (in practice, they usually point to some part of a `t_REAL` mantissa): they are considered as arrays of words representing some mantissa, and we shift globally `s` by $n > 0$ bits, storing the result in `t`. We assume that $m \leq M$ and only access `s[m], s[m + 1], \dots s[M]` (read) and likewise for `t` (write); we may have `s = t` but more general overlaps are not allowed. The word `f` is concatenated to `s` to supply extra bits.

`void shift_left(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

`s[m], s[m + 1], \dots s[M], f`

left by `n` bits.

`void shift_right(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

`f, s[m], s[m + 1], \dots s[M]`

right by `n` bits.

6.2.7 From `t_INT` to bits or digits in base 2^k and back.

GEN `binary_zv`(GEN `x`) given a `t_INT` `x`, return a `t_VEC` of bits, from most significant to least significant.

GEN `binary_2k`(GEN `x`, long `k`) given a `t_INT` `x`, and $k > 0$, return a `t_VEC` of digits of `x` in base 2^k , as `t_INT`s, from most significant to least significant.

GEN `binary_2k_nv`(GEN `x`, long `k`) given a `t_INT` `x`, and $0 < k < \text{BITS_IN_LONG}$, return a `t_VEC` of digits of `x` in base 2^k , as `ulong`s, from most significant to least significant.

GEN `bits_to_int`(GEN `x`, long `l`) given a vector `x` of `l` bits (as a `t_VEC` or even a pointer to a part of a larger vector, so not a proper GEN), return the integer $\sum_{i=1}^l x[i]2^{l-i}$, as a `t_INT`.

`ulong bits_to_u`(GEN `v`, long `l`) same as `bits_to_int`, where $l < \text{BITS_IN_LONG}$, so we can return an `ulong`.

GEN `fromdigitsu`(GEN `x`, GEN `B`) given a `t_VEC` `x` of length `l` and a `t_INT` `B`, return the integer $\sum_{i=1}^l x[i]B^{i-1}$, as a `t_INT`, where the `x[i]` are seen as unsigned integers.

GEN `fromdigits_2k`(GEN `x`, long `k`) converse of `binary_2k`; given a `t_VEC` `x` of length `l` and a positive long `k`, where each `x[i]` is a `t_INT` with $0 \leq x[i] < 2^k$, return the integer $\sum_{i=1}^l x[i]2^{k(l-i)}$, as a `t_INT`.

GEN `nv_fromdigits_2k`(GEN `x`, long `k`) as `fromdigits_2k`, but with `x` being a `t_VEC` and each `x[i]` being a `ulong` with $0 \leq x[i] < 2^{\min\{k, \text{BITS_IN_LONG}\}}$. Here `k` may be any positive long, and the `x[i]` are regarded as `k`-bit integers by truncating or extending with zeroes.

6.2.8 Integer valuation. For integers `x` and `p`, such that $x \neq 0$ and $|p| > 1$, we define $v_p(x)$ to be the largest integer exponent `e` such that p^e divides `x`. If `p` is prime, this is the ordinary valuation of `x` at `p`.

`long Z_pvalrem`(GEN `x`, GEN `p`, GEN `*r`) applied to `t_INT`s $x \neq 0$ and `p`, $|p| > 1$, returns $e := v_p(x)$. The quotient x/p^e is returned in `*r`. If `p` is a prime, `*r` is the prime-to-`p` part of `x`.

`long Z_pval`(GEN `x`, GEN `p`) as `Z_pvalrem` but only returns $v_p(x)$.

`long Z_lvalrem`(GEN `x`, `ulong p`, GEN `*r`) as `Z_pvalrem`, except that `p` is an `ulong` ($p > 1$).

`long Z_lvalrem_stop`(GEN `*x`, `ulong p`, `int *stop`) assume $x > 0$; returns $e := v_p(x)$ and replaces `x` by x/p^e . Set `stop` to 1 if the new value of `x` is $< p^2$ (and 0 otherwise). To be used when trial dividing `x` by successive primes: the `stop` condition is cheaply tested while testing whether `p` divides `x` (is the quotient less than `p`?), and allows to decide that `n` is prime if no prime $< p$ divides `n`. Not memory-clean.

`long Z_lval`(GEN `x`, `ulong p`) as `Z_pval`, except that `p` is an `ulong` ($p > 1$).

`long u_lvalrem`(`ulong x`, `ulong p`, `ulong *r`) as `Z_pvalrem`, except the inputs/outputs are now `ulong`s.

`long u_lvalrem_stop`(`ulong *n`, `ulong p`, `int *stop`) as `Z_pvalrem_stop`.

`long u_pvalrem`(`ulong x`, GEN `p`, `ulong *r`) as `Z_pvalrem`, except `x` and `r` are now `ulong`s.

`long u_lval`(`ulong x`, `ulong p`) as `Z_pval`, except the inputs are now `ulong`s.

`long u_pval`(`ulong x`, GEN `p`) as `Z_pval`, except `x` is now an `ulong`.

`long z_lval(long x, ulong p)` as `u_lval`, for signed `x`.
`long z_lvalrem(long x, ulong p)` as `u_lvalrem`, for signed `x`.
`long z_pval(long x, GEN p)` as `Z_pval`, except `x` is now a `long`.
`long z_pvalrem(long x, GEN p)` as `Z_pvalrem`, except `x` is now a `long`.
`long Q_pval(GEN x, GEN p)` valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.
`long factorial_lval(ulong n, ulong p)` returns $v_p(n!)$, assuming p is prime.

The following convenience functions generalize `Z_pval` and its variants to “containers” (`ZV` and `ZX`):

`long ZV_pvalrem(GEN x, GEN p, GEN *r)` x being a `ZV` (a vector of `t_INT`s), return the min v of the valuations of its components and set `*r` to x/p^v . Infinite loop if x is the zero vector. This function is not stack clean.

`long ZV_pval(GEN x, GEN p)` as `ZV_pvalrem` but only returns the “valuation”.

`int ZV_Z_dvd(GEN x, GEN p)` returns 1 if p divides all components of x and 0 otherwise. Faster than testing `ZV_pval(x,p) >= 1`.

`long ZV_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_pvalrem`, except that `p` is an `ulong` ($p > 1$). This function is not stack-clean.

`long ZV_lval(GEN x, ulong p)` as `ZV_pval`, except that `p` is an `ulong` ($p > 1$).

`long ZX_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZX` x (a `t_POL` with `t_INT` coefficients). This function is not stack-clean.

`long ZX_pval(GEN x, GEN p)` as `ZV_pval` for a `ZX` x .

`long ZX_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_lvalrem`, a `ZX` x . This function is not stack-clean.

`long ZX_lval(GEN x, ulong p)` as `ZX_pval`, except that `p` is an `ulong` ($p > 1$).

6.2.9 Generic unary operators. Let “*op*” be a unary operation among

- **neg**: negation ($-x$).
- **abs**: absolute value ($|x|$).
- **sqr**: square (x^2).

The names and prototypes of the low-level functions corresponding to *op* are as follows. The result is of the same type as `x`.

`GEN opi(GEN x)` creates the result of *op* applied to the `t_INT` `x`.

`GEN opr(GEN x)` creates the result of *op* applied to the `t_REAL` `x`.

`GEN mpop(GEN x)` creates the result of *op* applied to the `t_INT` or `t_REAL` `x`.

Complete list of available functions:

`GEN absi(GEN x), GEN absr(GEN x), GEN mpabs(GEN x)`

`GEN negi(GEN x), GEN negr(GEN x), GEN mpneg(GEN x)`

GEN sqri(GEN x), GEN sqrr(GEN x), GEN mpsqr(GEN x)

GEN absi_shallow(GEN x) x being a t_INT, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and $\text{negi}(x)$ otherwise.

GEN mpabs_shallow(GEN x) x being a t_INT or a t_REAL, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and $\text{mpneg}(x)$ otherwise.

Some miscellaneous routines:

GEN sqrs(long x) returns x^2 .

GEN sqru(ulong x) returns x^2 .

6.2.10 Comparison operators.

long minss(long x, long y)

ulong minuu(ulong x, ulong y)

double mindd(double x, double y) returns the min of x and y .

long maxss(long x, long y)

ulong maxuu(ulong x, ulong y)

double maxdd(double x, double y) returns the max of x and y .

int mpcmp(GEN x, GEN y) compares the t_INT or t_REAL x to the t_INT or t_REAL y . The result is the sign of $x - y$.

int cmpii(GEN x, GEN y) compares the t_INT x to the t_INT y .

int cmpir(GEN x, GEN y) compares the t_INT x to the t_REAL y .

int cmpis(GEN x, long s) compares the t_INT x to the long s .

int cmpiu(GEN x, ulong s) compares the t_INT x to the ulong s .

int cmpsi(long s, GEN x) compares the long s to the t_INT x .

int cmpui(ulong s, GEN x) compares the ulong s to the t_INT x .

int cmpsr(long s, GEN x) compares the long s to the t_REAL x .

int cmpri(GEN x, GEN y) compares the t_REAL x to the t_INT y .

int cmprr(GEN x, GEN y) compares the t_REAL x to the t_REAL y .

int cmprs(GEN x, long s) compares the t_REAL x to the long s .

int equalii(GEN x, GEN y) compares the t_INTs x and y . The result is 1 if $x = y$, 0 otherwise.

int equalrr(GEN x, GEN y) compares the t_REALs x and y . The result is 1 if $x = y$, 0 otherwise. Equality is decided according to the following rules: all real zeroes are equal, and different from a non-zero real; two non-zero reals are equal if all their digits coincide up to the length of the shortest of the two, and the remaining words in the mantissa of the longest are all 0.

int equalis(GEN x, long s) compare the t_INT x and the long s . The result is 1 if $x = y$, 0 otherwise.

int equalsi(long s, GEN x)

`int equaliu(GEN x, ulong s)` compare the `t_INT` `x` and the `ulong` `s`. The result is 1 if `x = y`, 0 otherwise.

`int equalui(ulong s, GEN x)`

The remaining comparison operators disregard the sign of their operands

`int absequaliu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`. The result is 1 if `|x| = y`, 0 otherwise. This is marginally more efficient than `equalis` even when `x` is known to be non-negative.

`int absequalui(ulong u, GEN x)`

`int absncmpiu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `u`.

`int absncmpui(ulong u, GEN x)`

`int absncmpii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is the sign of `|x| - |y|`.

`int absequalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if `|x| = |y|`, 0 otherwise.

`int absncmprr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is the sign of `|x| - |y|`.

`int absrnz_equal2n(GEN x)` tests whether a non-zero `t_REAL` `x` is equal to $\pm 2^e$ for some integer `e`.

`int absrnz_equal1(GEN x)` tests whether a non-zero `t_REAL` `x` is equal to ± 1 .

6.2.11 Generic binary operators. The operators in this section have arguments of C-type `GEN`, `long`, and `ulong`, and only `t_INT` and `t_REAL` `GEN`s are allowed. We say an argument is a real type if it is a `t_REAL` `GEN`, and an integer type otherwise. The result is always a `t_REAL` unless both `x` and `y` are integer types.

Let “*op*” be a binary operation among

- **add**: addition (`x + y`).
- **sub**: subtraction (`x - y`).
- **mul**: multiplication (`x * y`).
- **div**: division (`x / y`). In the case where `x` and `y` are both integer types, the result is the Euclidean quotient, where the remainder has the same sign as the dividend `x`. It is the ordinary division otherwise. A division-by-0 error occurs if `y` is equal to 0.

The last two generic operations are defined only when arguments have integer types; and the result is a `t_INT`:

- **rem**: remainder (“`x % y`”). The result is the Euclidean remainder corresponding to **div**, i.e. its sign is that of the dividend `x`.

- **mod**: true remainder (`x % y`). The result is the true Euclidean remainder, i.e. non-negative and less than the absolute value of `y`.

Important technical note. The rules given above fixing the output type (to `t_REAL` unless both inputs are integer types) are subtly incompatible with the general rules obeyed by PARI's generic functions, such as `gmul` or `gdiv` for instance: the latter return a result containing as much information as could be deduced from the inputs, so it is not true that if x is a `t_INT` and y a `t_REAL`, then `gmul(x,y)` is always the same as `mulir(x,y)`. The exception is $x = 0$, in that case we can deduce that the result is an exact 0, so `gmul` returns `gen_0`, while `mulir` returns a `t_REAL 0`. Specifically, the one resulting from the conversion of `gen_0` to a `t_REAL` of precision `precision(y)`, multiplied by y ; this determines the exponent of the real 0 we obtain.

The reason for the discrepancy between the two rules is that we use the two sets of functions in different contexts: generic functions allow to write high-level code forgetting about types, letting PARI return results which are sensible and as simple as possible; type specific functions are used in kernel programming, where we do care about types and need to maintain strict consistency: it is much easier to compute the types of results when they are determined from the types of the inputs only (without taking into account further arithmetic properties, like being non-0).

The names and prototypes of the low-level functions corresponding to *op* are as follows. In this section, the `z` argument in the `z`-functions must be of type `t_INT` when no `r` or `mp` appears in the argument code (no `t_REAL` operand is involved, only integer types), and of type `t_REAL` otherwise.

`GEN mpop[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT` or `t_REAL` x and y . The function `mpdivz` does not exist (its semantic would change drastically depending on the type of the `z` argument), and neither do `mprem[z]` nor `mpmod[z]` (specific to integers).

`GEN opsi[z](long s, GEN x[, GEN z])` applies *op* to the `long` s and the `t_INT` x . These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opsr[z](long s, GEN x[, GEN z])` applies *op* to the `long` s and the `t_REAL` x .

`GEN opss[z](long s, long t[, GEN z])` applies *op* to the `long`s s and t . These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opii[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT`s x and y . These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opir[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT` x and the `t_REAL` y .

`GEN opis[z](GEN x, long s[, GEN z])` applies *op* to the `t_INT` x and the `long` s . These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opri[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_REAL` x and the `t_INT` y .

`GEN oprr[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_REAL`s x and y .

`GEN oprs[z](GEN x, long s[, GEN z])` applies *op* to the `t_REAL` x and the `long` s .

Some miscellaneous routines:

`long expu(ulong x)` assuming $x > 0$, returns the binary exponent of the real number equal to x . This is a special case of `gexpo`.

`GEN adduu(ulong x, ulong y)`

`GEN addiu(GEN x, ulong y)`

`GEN addui(ulong x, GEN y)` adds x and y .

`GEN subuu(ulong x, ulong y)`

GEN subiu(GEN x, ulong y)
 GEN subui(ulong x, GEN y) subtracts x by y.
 GEN muluu(ulong x, ulong y) multiplies x by y.
 ulong umuluu_le(ulong x, ulong y, ulong n) multiplies x by y. Return xy if $xy \leq n$ and 0 otherwise (in particular if xy does not fit in an ulong).
 ulong umuluu_or_0(ulong x, ulong y) multiplies x by y. Return 0 if xy does not fit in an ulong.
 GEN mului(ulong x, GEN y) multiplies x by y.
 GEN muluui(ulong x, ulong y, GEN z) return xyz .
 GEN muliu(GEN x, ulong y) multiplies x by y.
 void addumului(ulong a, ulong b, GEN x) return $a + b|X|$.
 GEN addmuliu(GEN x, GEN y, ulong u) returns $x + yu$.
 GEN addmulii(GEN x, GEN y, GEN z) returns $x + yz$.
 GEN addmulii_inplace(GEN x, GEN y, GEN z) returns $x + yz$, but returns x itself and not a copy if $yz = 0$. Not suitable for gerepile or gerepileupto.
 GEN addmuliu_inplace(GEN x, GEN y, ulong u) returns $x + yu$, but returns x itself and not a copy if $yu = 0$. Not suitable for gerepile or gerepileupto.
 GEN submuliu_inplace(GEN x, GEN y, ulong u) returns $x - yu$, but returns x itself and not a copy if $yu = 0$. Not suitable for gerepile or gerepileupto.
 GEN lincombii(GEN u, GEN v, GEN x, GEN y) returns $ux + vy$.
 GEN mulsubii(GEN y, GEN z, GEN x) returns $yz - x$.
 GEN submulii(GEN x, GEN y, GEN z) returns $x - yz$.
 GEN submuliu(GEN x, GEN y, ulong u) returns $x - yu$.
 GEN mulu_interval(ulong a, ulong b) returns $a(a + 1) \cdots b$, assuming that $a \leq b$.
 GEN muls_interval(long a, long b) returns $a(a + 1) \cdots b$, assuming that $a \leq b$.
 GEN invr(GEN x) returns the inverse of the non-zero t_REAL x .
 GEN truedivii(GEN x, GEN y) returns the true Euclidean quotient (with non-negative remainder less than $|y|$).
 GEN truedivis(GEN x, long y) returns the true Euclidean quotient (with non-negative remainder less than $|y|$).
 GEN truedivsi(long x, GEN y) returns the true Euclidean quotient (with non-negative remainder less than $|y|$).
 GEN centermodii(GEN x, GEN y, GEN y2), given t_INTs x, y , returns z congruent to x modulo y , such that $-y/2 \leq z < y/2$. The function requires an extra argument $y2$, such that $y2 = \text{shifti}(y, -1)$. (In most cases, y is constant for many reductions and $y2$ need only be computed once.)
 GEN remi2n(GEN x, long n) returns $x \bmod 2^n$.

GEN addii_sign(GEN x, long sx, GEN y, long sy) add the t_INT s x and y as if their signs were sx and sy .

GEN addir_sign(GEN x, long sx, GEN y, long sy) add the t_INT x and the t_REAL y as if their signs were sx and sy .

GEN addrr_sign(GEN x, long sx, GEN y, long sy) add the t_REAL s x and y as if their signs were sx and sy .

GEN addsi_sign(long x, GEN y, long sy) add x and the t_INT y as if its sign was sy .

GEN addui_sign(ulong x, GEN y, long sy) add x and the t_INT y as if its sign was sy .

6.2.12 Exact division and divisibility.

GEN diviexact(GEN x, GEN y) returns the Euclidean quotient x/y , assuming y divides x . Uses Jebelean algorithm (Jebelean-Krandick bidirectional exact division is not implemented).

GEN diviuexact(GEN x, ulong y) returns the Euclidean quotient x/y , assuming y divides x and y is non-zero.

GEN diviuuexact(GEN x, ulong y, ulong z) returns the Euclidean quotient $x/(yz)$, assuming yz divides x and $yz \neq 0$.

The following routines return 1 (true) if y divides x , and 0 otherwise. (Error if y is 0, even if x is 0.) All GEN are assumed to be t_INT s:

```
int dvdi(GEN x, GEN y), int dvdis(GEN x, long y), int dvdiu(GEN x, ulong y),
int dvdsi(long x, GEN y), int dvdui(ulong x, GEN y).
```

The following routines return 1 (true) if y divides x , and in that case assign the quotient to z ; otherwise they return 0. All GEN are assumed to be t_INT s:

```
int dvdiiz(GEN x, GEN y, GEN z), int dvdisz(GEN x, long y, GEN z).
int dvdiuz(GEN x, ulong y, GEN z) if  $y$  divides  $x$ , assigns the quotient  $|x|/y$  to  $z$  and returns 1 (true), otherwise returns 0 (false).
```

6.2.13 Division with integral operands and t_REAL result.

GEN rdivii(GEN x, GEN y, long prec), assuming x and y are both of type t_INT , return the quotient x/y as a t_REAL of precision $prec$.

GEN rdiviiz(GEN x, GEN y, GEN z), assuming x and y are both of type t_INT , and z is a t_REAL , assign the quotient x/y to z .

GEN rdivis(GEN x, long y, long prec), assuming x is of type t_INT , return the quotient x/y as a t_REAL of precision $prec$.

GEN rdivsi(long x, GEN y, long prec), assuming y is of type t_INT , return the quotient x/y as a t_REAL of precision $prec$.

GEN rdivss(long x, long y, long prec), return the quotient x/y as a t_REAL of precision $prec$.

6.2.14 Division with remainder. The following functions return two objects, unless specifically asked for only one of them — a quotient and a remainder. The quotient is returned and the remainder is returned through the variable whose address is passed as the `r` argument. The term *true Euclidean remainder* refers to the non-negative one (`mod`), and *Euclidean remainder* by itself to the one with the same sign as the dividend (`rem`). All GENs, whether returned directly or through a pointer, are created on the stack.

GEN `dvmdii`(GEN `x`, GEN `y`, GEN `*r`) returns the Euclidean quotient of the `t_INT` `x` by a `t_INT` `y` and puts the remainder into `*r`. If `r` is equal to `NULL`, the remainder is not created, and if `r` is equal to `ONLY_REM`, only the remainder is created and returned. In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`. The remainder is always of the sign of the dividend `x`. If the remainder is 0 set `r = gen_0`.

void `dvmdiiz`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) assigns the Euclidean quotient of the `t_INT`s `x` and `y` into the `t_INT` `z`, and the Euclidean remainder into the `t_INT` `t`.

Analogous routines `dvmdis[z]`, `dvmdsi[z]`, `dvmdss[z]` are available, where `s` denotes a `long` argument. But the following routines are in general more flexible:

long `sdivss_rem`(long `s`, long `t`, long `*r`) computes the Euclidean quotient and remainder of the longs `s` and `t`. Puts the remainder into `*r`, and returns the quotient. The remainder is of the sign of the dividend `s`, and has strictly smaller absolute value than `t`.

long `sdivsi_rem`(long `s`, GEN `x`, long `*r`) computes the Euclidean quotient and remainder of the long `s` by the `t_INT` `x`. As `sdivss_rem` otherwise.

long `sdivsi`(long `s`, GEN `x`) as `sdivsi_rem`, without remainder.

GEN `divis_rem`(GEN `x`, long `s`, long `*r`) computes the Euclidean quotient and remainder of the `t_INT` `x` by the long `s`. As `sdivss_rem` otherwise.

GEN `absdiviu_rem`(GEN `x`, ulong `s`, ulong `*r`) computes the Euclidean quotient and remainder of *absolute value* of the `t_INT` `x` by the ulong `s`. As `sdivss_rem` otherwise.

ulong `uabsdiviu_rem`(GEN `n`, ulong `d`, ulong `*r`) as `absdiviu_rem`, assuming that $|n|/d$ fits into an ulong.

ulong `uabsdivui_rem`(ulong `x`, GEN `y`, ulong `*rem`) computes the Euclidean quotient and remainder of `x` by $|y|$. As `sdivss_rem` otherwise.

ulong `udivuu_rem`(ulong `x`, ulong `y`, ulong `*rem`) computes the Euclidean quotient and remainder of `x` by `y`. As `sdivss_rem` otherwise.

ulong `ceildivuu`(ulong `x`, ulong `y`) return the ceiling of x/y .

GEN `divsi_rem`(long `s`, GEN `y`, long `*r`) computes the Euclidean quotient and remainder of the long `s` by the GEN `y`. As `sdivss_rem` otherwise.

GEN `divss_rem`(long `x`, long `y`, long `*r`) computes the Euclidean quotient and remainder of the long `x` by the long `y`. As `sdivss_rem` otherwise.

GEN `truedvmdii`(GEN `x`, GEN `y`, GEN `*r`), as `dvmdii` but with a non-negative remainder.

GEN `truedvmdis`(GEN `x`, long `y`, GEN `*z`), as `dvmdis` but with a non-negative remainder.

GEN `truedvmdsi`(long `x`, GEN `y`, GEN `*z`), as `dvmdsi` but with a non-negative remainder.

6.2.15 Modulo to longs. The following variants of `modii` do not clutter the stack:

`long smodis(GEN x, long y)` computes the true Euclidean remainder of the `t_INT` `x` by the `long` `y`. This is the non-negative remainder, not the one whose sign is the sign of `x` as in the `div` functions.

`long smodss(long x, long y)` computes the true Euclidean remainder of the `long` `x` by a `long` `y`.

`ulong umodsu(long x, ulong y)` computes the true Euclidean remainder of the `long` `x` by a `ulong` `y`.

`ulong umodiu(GEN x, ulong y)` computes the true Euclidean remainder of the `t_INT` `x` by the `ulong` `y`.

`ulong umodui(ulong x, GEN y)` computes the true Euclidean remainder of the `ulong` `x` by the `t_INT` `|y|`.

The routine `smodsi` does not exist, since it would not always be defined: for a *negative* `x`, if the quotient is ± 1 , the result `x + |y|` would in general not fit into a `long`. Use either `umodui` or `modsi`.

These functions directly access the binary data and are thus much faster than the generic modulo functions:

`int mpodd(GEN x)` which is 1 if `x` is odd, and 0 otherwise.

`ulong Mod2(GEN x)`

`ulong Mod4(GEN x)`

`ulong Mod8(GEN x)`

`ulong Mod16(GEN x)`

`ulong Mod32(GEN x)`

`ulong Mod64(GEN x)` give the residue class of `x` modulo the corresponding power of 2.

`ulong umodi2n(GEN x, long n)` give the residue class of `x` modulo 2^n , $0 \leq n < BITS_IN_LONG$.

The following functions assume that $x \neq 0$ and in fact disregard the sign of `x`. There are about 10% faster than the safer variants above:

`long mod2(GEN x)`

`long mod4(GEN x)`

`long mod8(GEN x)`

`long mod16(GEN x)`

`long mod32(GEN x)`

`long mod64(GEN x)` give the residue class of $|x|$ modulo the corresponding power of 2, for *non-zero* `x`. As well,

`ulong mod2BIL(GEN x)` returns the least significant word of $|x|$, still assuming that $x \neq 0$.

6.2.16 Powering, Square root.

GEN `powii`(GEN `x`, GEN `n`), assumes x and n are `t_INT`s and returns x^n .

GEN `powuu`(ulong `x`, ulong `n`), returns x^n .

GEN `powiu`(GEN `x`, ulong `n`), assumes x is a `t_INT` and returns x^n .

GEN `powis`(GEN `x`, long `n`), assumes x is a `t_INT` and returns x^n (possibly a `t_FRAC` if $n < 0$).

GEN `powrs`(GEN `x`, long `n`), assumes x is a `t_REAL` and returns x^n . This is considered as a sequence of `mulrr`, possibly empty: as such the result has type `t_REAL`, even if $n = 0$. Note that the generic function `gpwgs(x,0)` would return `gen_1`, see the technical note in Section 6.2.11.

GEN `powru`(GEN `x`, ulong `n`), assumes x is a `t_REAL` and returns x^n (always a `t_REAL`, even if $n = 0$).

GEN `powersr`(GEN `e`, long `n`). Given a `t_REAL` e , return the vector v of all e^i , $0 \leq i \leq n$, where $v[i] = e^{i-1}$.

GEN `powrshalf`(GEN `x`, long `n`), assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

GEN `powruhalf`(GEN `x`, ulong `n`), assumes x is a `t_REAL` and returns $x^{n/2}$ (always a `t_REAL`, even if $n = 0$).

GEN `powfrac`(GEN `x`, long `n`, long `d`), assumes x is a `t_REAL` and returns $x^{n/d}$ (always a `t_REAL`, even if $n = 0$).

GEN `powIs`(long `n`) returns $I^n \in \{1, I, -1, -I\}$ (`t_INT` for even n , `t_COMPLEX` otherwise).

ulong `upowuu`(ulong `x`, ulong `n`), returns x^n when $< 2^{\text{BITS_IN_LONG}}$, and 0 otherwise (overflow).

GEN `sqrtremi`(GEN `N`, GEN `*r`), returns the integer square root S of the non-negative `t_INT` N (rounded towards 0) and puts the remainder R into `*r`. Precisely, $N = S^2 + R$ with $0 \leq R \leq 2S$. If `r` is equal to `NULL`, the remainder is not created. In the generic case, the remainder is created after the quotient and can be disposed of individually with `cgiv(R)`. If the remainder is 0 set `R = gen_0`.

Uses a divide and conquer algorithm (discrete variant of Newton iteration) due to Paul Zimmermann (“Karatsuba Square Root”, INRIA Research Report 3805 (1999)).

GEN `sqrti`(GEN `N`), returns the integer square root S of the non-negative `t_INT` N (rounded towards 0). This is identical to `sqrtremi(N, NULL)`.

long `logintall`(GEN `B`, GEN `y`, GEN `*ptq`) returns the floor e of $\log_y B$, where $B > 0$ and $y > 1$ are integers. If `ptq` is not `NULL`, set it to y^e . (Analogous to `logint0`, without sanity checks.)

ulong `ulongintall`(ulong `B`, ulong `y`, ulong `*ptq`) as `logintall` for `ulong` arguments.

long `logint`(GEN `B`, GEN `y`) returns the floor e of $\log_y B$, where $B > 0$ and $y > 1$ are integers.

ulong `ulongint`(ulong `B`, ulong `y`) as `logint` for `ulong` arguments.

GEN `vecpowuu`(long `N`, ulong `a`) return the vector of n^a , $n = 1, \dots, N$. Not memory clean.

GEN `vecpowug`(long `N`, GEN `a`, long `prec`) return the vector of n^a , $n = 1, \dots, N$, where the powers are computed at precision `prec`. Not memory clean.

6.2.17 GCD, extended GCD and LCM.

`long cgcd(long x, long y)` returns the GCD of `x` and `y`.

`ulong ugcd(ulong x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdiu(GEN x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdui(ulong x, GEN y)` returns the GCD of `x` and `y`.

`GEN coprimes_zv(ulong N)` return a `t_VECSMALL` `T` with `N` entries such that $T[i] = 1$ iff $(i, N) = 1$ and 0 otherwise.

`long clcm(long x, long y)` returns the LCM of `x` and `y`, provided it fits into a `long`. Silently overflows otherwise.

`ulong ulcm(ulong x, ulong y)` returns the LCM of `x` and `y`, provided it fits into an `ulong`. Silently overflows otherwise.

`GEN gcdii(GEN x, GEN y)`, returns the GCD of the `t_INTs` `x` and `y`.

`GEN lcmii(GEN x, GEN y)`, returns the LCM of the `t_INTs` `x` and `y`.

`GEN bezout(GEN a, GEN b, GEN *u, GEN *v)`, returns the GCD d of `t_INTs` `a` and `b` and sets `u`, `v` to the Bezout coefficients such that $au + bv = d$.

`long cbezout(long a, long b, long *u, long *v)`, returns the GCD d of `a` and `b` and sets `u`, `v` to the Bezout coefficients such that $au + bv = d$.

`GEN ZV_extgcd(GEN A)` given a vector of n integers `A`, returns $[d, U]$, where d is the GCD of the $A[i]$ and U is a matrix in $GL_n(\mathbf{Z})$ such that $AU = [0, \dots, 0, D]$.

6.2.18 Continued fractions and convergents.

`GEN ZV_allpnqn(GEN x)` given $x = [a_0, \dots, a_n]$ a continued fraction from `gboundcf`, $n \geq 0$, return all convergents as $[P, Q]$, where $P = [p_0, \dots, p_n]$ and $Q = [q_0, \dots, q_n]$.

6.2.19 Pseudo-random integers. These routine return pseudo-random integers uniformly distributed in some interval. They all use the same underlying generator which can be seeded and restarted using `getrand` and `setrand`.

`void setrand(GEN seed)` reseeds the random number generator using the seed `n`. The seed is either a technical array output by `getrand` or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by `setrand(1)` twice will generate the exact same output.

`GEN getrand(void)` returns the current value of the seed used by the pseudo-random number generator `random`. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array of type `t_VECSMALL`), and can only be used as an argument to `setrand`.

`ulong pari_rand(void)` returns a random $0 \leq x < 2^{\text{BITS_IN_LONG}}$.

`long random_bits(long k)` returns a random $0 \leq x < 2^k$. Assumes that $0 \leq k \leq \text{BITS_IN_LONG}$.

`ulong random_Fl(ulong p)` returns a pseudo-random integer in $0, 1, \dots, p - 1$.

`GEN randomi(GEN n)` returns a random `t_INT` between 0 and `n - 1`.

`GEN randomr(long prec)` returns a random `t_REAL` in $[0, 1[$, with precision `prec`.

6.2.20 Modular operations. In this subsection, all GENs are `t_INT`.

`GEN Fp_red(GEN a, GEN m)` returns a modulo m (smallest non-negative residue). (This is identical to `modii`).

`GEN Fp_neg(GEN a, GEN m)` returns $-a$ modulo m (smallest non-negative residue).

`GEN Fp_add(GEN a, GEN b, GEN m)` returns the sum of a and b modulo m (smallest non-negative residue).

`GEN Fp_sub(GEN a, GEN b, GEN m)` returns the difference of a and b modulo m (smallest non-negative residue).

`GEN Fp_center(GEN a, GEN p, GEN pov2)` assuming that `pov2` is `shifti(p,-1)` and that $-p/2 < a < p$, returns the representative of a in the symmetric residue system $]-p/2, p/2]$.

`GEN Fp_center_i(GEN a, GEN p, GEN pov2)` internal variant of `Fp_center`, not `gerepile-safe`: when a is already in the proper interval, it is returned as is, without a copy.

`GEN Fp_mul(GEN a, GEN b, GEN m)` returns the product of a by b modulo m (smallest non-negative residue).

`GEN Fp_addmul(GEN x, GEN y, GEN z, GEN p)` returns $x + yz$.

`GEN Fp_mulu(GEN a, ulong b, GEN m)` returns the product of a by b modulo m (smallest non-negative residue).

`GEN Fp_muls(GEN a, long b, GEN m)` returns the product of a by b modulo m (smallest non-negative residue).

`GEN Fp_half(GEN x, GEN m)` returns z such that $2z = x$ modulo m assuming such z exists.

`GEN Fp_sqr(GEN a, GEN m)` returns a^2 modulo m (smallest non-negative residue).

`ulong Fp_powu(GEN x, ulong n, GEN m)` raises x to the n -th power modulo m (smallest non-negative residue). Not memory-clean, but suitable for `gerepileupto`.

`ulong Fp_pows(GEN x, long n, GEN m)` raises x to the n -th power modulo m (smallest non-negative residue). A negative n is allowed. Not memory-clean, but suitable for `gerepileupto`.

`GEN Fp_pow(GEN x, GEN n, GEN m)` returns x^n modulo m (smallest non-negative residue).

`GEN Fp_pow_init(GEN x, GEN n, long k, GEN p)` Return a table R that can be used with `Fp_pow_table` to compute the powers of x up to n . The table is of size $2^k \log_2(n)$.

`GEN Fp_pow_table(GEN R, GEN n, GEN p)` return x^n , where R is given by `Fp_pow_init(x,m,k,p)` for some integer $m \geq n$.

`GEN Fp_powers(GEN x, long n, GEN m)` returns $[x^0, \dots, x^n]$ modulo m as a `t_VEC` (smallest non-negative residue).

`GEN Fp_inv(GEN a, GEN m)` returns an inverse of a modulo m (smallest non-negative residue). Raise an error if a is not invertible.

`GEN Fp_invsafe(GEN a, GEN m)` as `Fp_inv`, but return `NULL` if a is not invertible.

`GEN Fp_invgen(GEN x, GEN m, GEN *pg)` set `*pg` to $g = \gcd(x, m)$ and return u in $(\mathbf{Z}/m\mathbf{Z})^*$ such that $xu = g$ modulo m . We have $g = 1$ if and only if x is invertible, and in this case u is its inverse.

`GEN FpV_inv(GEN x, GEN m)` x being a vector of `t_INTs`, return the vector of inverses of the $x[i] \bmod m$. The routine uses Montgomery's trick, and involves a single inversion mod m , plus $3(N-1)$

multiplications for N entries. The routine is not stack-clean: $2N$ integers mod m are left on stack, besides the N in the result.

`GEN Fp_div(GEN a, GEN b, GEN m)` returns the quotient of a by b modulo m (smallest non-negative residue). Raise an error if b is not invertible.

`int invmod(GEN a, GEN m, GEN *g)`, return 1 if a modulo m is invertible, else return 0 and set $g = \gcd(a, m)$.

In the following three functions the integer parameter `ord` can be given either as a positive `t_INT` N , or as its factorization matrix faN , or as a pair $[N, faN]$. The parameter may be omitted by setting it to `NULL` (the value is then $p - 1$).

`GEN Fp_log(GEN a, GEN g, GEN ord, GEN p)` Let g such that $g^{ord} \equiv 1 \pmod{p}$. Return an integer e such that $a^e \equiv g \pmod{p}$. If e does not exist, the result is undefined.

`GEN Fp_order(GEN a, GEN ord, GEN p)` returns the order of the `Fp` a . Assume that `ord` is a multiple of the order of a .

`GEN Fp_factored_order(GEN a, GEN ord, GEN p)` returns $[o, F]$, where o is the multiplicative order of the `Fp` a in \mathbf{F}_p^* , and F is the factorization of o . Assume that `ord` is a multiple of the order of a .

`int Fp_issquare(GEN x, GEN p)` returns 1 if x is a square modulo p , and 0 otherwise.

`int Fp_ispower(GEN x, GEN n, GEN p)` returns 1 if x is an n -th power modulo p , and 0 otherwise.

`GEN Fp_sqrt(GEN x, GEN p)` returns a square root of x modulo p (the smallest non-negative residue), where x, p are `t_INT`s, and p is assumed to be prime. Return `NULL` if x is not a quadratic residue modulo p .

`GEN Fp_2gener(GEN p)` return a generator of the 2-Sylow subgroup of \mathbf{F}_p^* . To use with `Fp_sqrt_i`.

`GEN Fp_sqrt_i(GEN x, GEN s2, GEN p)` as `Fp_sqrt` where `s2` is the element returned by `Fp_2gener`.

`GEN Fp_sqrtn(GEN a, GEN n, GEN p, GEN *zn)` returns `NULL` if a is not an n -th power residue mod p . Otherwise, returns an n -th root of a ; if `zn` is non-`NULL` set it to a primitive m -th root of 1, $m = \gcd(p - 1, n)$ allowing to compute all m solutions in \mathbf{F}_p of the equation $x^n = a$.

`GEN Zn_sqrt(GEN x, GEN n)` returns one of the square roots of x modulo n (possibly not prime), where x is a `t_INT` and n is either a `t_INT` or is given by its factorization matrix. Return `NULL` if no such square root exist.

`long kross(long x, long y)` returns the Kronecker symbol $(x|y)$, i.e. $-1, 0$ or 1 . If y is an odd prime, this is the Legendre symbol. (Contrary to `krouu`, `kross` also supports $y = 0$)

`long krouu(ulong x, ulong y)` returns the Kronecker symbol $(x|y)$, i.e. $-1, 0$ or 1 . Assumes y is non-zero. If y is an odd prime, this is the Legendre symbol.

`long krois(GEN x, long y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and `long` y . As `kross` otherwise.

`long krouu(GEN x, ulong y)` returns the Kronecker symbol $(x|y)$ of `t_INT` x and non-zero `ulong` y . As `krouu` otherwise.

`long krosi(long x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT` y . As `kross` otherwise.

`long kroui(ulong x, GEN y)` returns the Kronecker symbol $(x|y)$ of `long` x and `t_INT` y . As `kross` otherwise.

`long kronecker(GEN x, GEN y)` returns the Kronecker symbol $(x|y)$ of `t_INTs` x and y . As `kross` otherwise.

`GEN pgener_Fp(GEN p)` returns the smallest primitive root modulo p , assuming p is prime.

`GEN pgener_Zp(GEN p)` returns the smallest primitive root modulo p^k , $k > 1$, assuming p is an odd prime.

`long Zp_issquare(GEN x, GEN p)` returns 1 if the `t_INT` x is a p -adic square, 0 otherwise.

`long Zn_issquare(GEN x, GEN n)` returns 1 if `t_INT` x is a square modulo n (possibly not prime), where n is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise.

`long Zn_isplayer(GEN x, GEN n, GEN K, GEN *py)` returns 1 if `t_INT` x is a K -th power modulo n (possibly not prime), where n is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise. If `py` is not `NULL`, set it to y such that $y^K = x$ modulo n .

`GEN pgener_Fp_local(GEN p, GEN L)`, L being a vector of primes dividing $p - 1$, returns the smallest integer $x > 1$ which is a generator of the ℓ -Sylow of \mathbf{F}_p^* for every ℓ in L . In other words, $x^{(p-1)/\ell} \neq 1$ for all such ℓ . In particular, returns `pgener_Fp(p)` if L contains all primes dividing $p - 1$. It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square.

`GEN rootsof1_Fp(GEN n, GEN p)` returns a primitive n -th root modulo the prime p .

`GEN rootsof1u_Fp(ulong n, GEN p)` returns a primitive n -th root modulo the prime p .

`ulong rootsof1_Fl(ulong n, ulong p)` returns a primitive n -th root modulo the prime p .

6.2.21 Extending functions to vector inputs.

The following functions apply f to the given arguments, recursively if they are of vector / matrix type:

`GEN map_proto_G(GEN (*f)(GEN), GEN x)` For instance, if x is a `t_VEC`, return a `t_VEC` whose components are the $f(x[i])$.

`GEN map_proto_lG(long (*f)(GEN), GEN x)` As above, applying the function `stoi(f())`.

`GEN map_proto_GL(GEN (*f)(GEN, long), GEN x, long y)`

`GEN map_proto_lGL(long (*f)(GEN, long), GEN x, long y)`

In the last function, f implements an associative binary operator, which we extend naturally to an n -ary operator f_n for any n : by convention, $f_0() = 1$, $f_1(x) = x$, and

$$f_n(x_1, \dots, x_n) = f(f_{n-1}(x_1, \dots, x_{n-1}), x_n),$$

for $n \geq 2$.

`GEN gassoc_proto(GEN (*f)(GEN, GEN), GEN x, GEN y)` If y is not `NULL`, return $f(x, y)$. Otherwise, x must be of vector type, and we return the result of f applied to its components, computed using a divide-and-conquer algorithm. More precisely, return

$$f(f(x_1, \text{NULL}), f(x_2, \text{NULL})),$$

where x_1, x_2 are the two halves of x .

6.2.22 Miscellaneous arithmetic functions.

`long bigomegau(ulong n)` returns the number of prime divisors of $n > 0$, counted with multiplicity.

`ulong coreu(ulong n)`, unique squarefree integer d dividing n such that n/d is a square.

`ulong coreu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`ulong corediscs(long d, ulong *pt_f)`, d (possibly negative) being congruent to 0 or 1 modulo 4, return the fundamental discriminant D such that $d = D * f^2$ and set `*pt_f` to f (if `*pt_f` not NULL).

`ulong eulerphiu(ulong n)`, Euler's totient function of n .

`ulong eulerphiu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long moebiusu(ulong n)`, Moebius μ -function of n .

`long moebiusu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`GEN divisorsu(ulong n)`, returns the divisors of n in a `t_VECSMALL`, sorted by increasing order.

`GEN divisorsu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long numdivu(ulong n)`, returns the number of positive divisors of $n > 0$.

`long numdivu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long omegau(ulong n)` returns the number of prime divisors of $n > 0$.

`long uissquarefree(ulong n)` returns 1 if n is square-free, and 0 otherwise.

`long uissquarefree_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long uposisfundamental(ulong x)` return 1 if x is a fundamental discriminant, and 0 otherwise.

`long unegisfundamental(ulong x)` return 1 if $-x$ is a fundamental discriminant, and 0 otherwise.

`long sisfundamental(long x)` return 1 if x is a fundamental discriminant, and 0 otherwise.

`int uis_357_power(ulong x, ulong *pt, ulong *mask)` as `is_357_power` for `ulong x`.

`int uis_357_powermod(ulong x, ulong *mask)` as `uis_357_power`, but only check for 3rd, 5th or 7th powers modulo $211 \times 209 \times 61 \times 203 \times 117 \times 31 \times 43 \times 71$.

`long uisprimepower(ulong n, ulong *p)` as `isprimepower`, for `ulong n`.

`int uislucaspsp(ulong n)` returns 1 if the `ulong n` fails Lucas compositeness test (it thus may be prime or composite), and 0 otherwise (proving that n is composite).

`ulong sumdigitu(ulong n)` returns the sum of decimal digits of u .

`GEN usumdiv_fact(GEN fa)`, sum of divisors of `ulong n`, where `fa` is `factoru(n)`.

`GEN usumdivk_fact(GEN fa, ulong k)`, sum of k -th powers of divisors of `ulong n`, where `fa` is `factoru(n)`.

`GEN hilbertii(GEN x, GEN y, GEN p)`, returns the Hilbert symbol (x, y) at the prime p (NULL for the place at infinity); x and y are `t_INTs`.

`GEN sumdedekind(GEN h, GEN k)` returns the Dedekind sum attached to the `t_INT` h and k , $k > 0$.

GEN `sumdedekind_coprime(GEN h, GEN k)` as `sumdedekind`, except that h and k are assumed to be coprime `t_INTs`.

GEN `u_sumdedekind_coprime(long h, long k)` Let $k > 0$, $0 \leq h < k$, $(h, k) = 1$. Returns $[s_1, s_2]$ in a `t_VECSMALL`, such that $s(h, k) = (s_2 + ks_1)/(12k)$. Requires $\max(h + k/2, k) < \text{LONG_MAX}$ to avoid overflow, in particular $k \leq (2/3)\text{LONG_MAX}$ is fine.

Chapter 7: Level 2 kernel

These functions deal with modular arithmetic, linear algebra and polynomials where assumptions can be made about the types of the coefficients.

7.1 Naming scheme.

A function name is built in the following way: $A_1 \dots A_n fun$ for an operation fun with n arguments of class A_1, \dots, A_n . A class name is given by a base ring followed by a number of code letters. Base rings are among

F1: $\mathbf{Z}/l\mathbf{Z}$ where $l < 2^{\text{BITS_IN_LONG}}$ is not necessarily prime. Implemented using `ulongs`

Fp: $\mathbf{Z}/p\mathbf{Z}$ where p is a `t_INT`, not necessarily prime. Implemented as `t_INTs` z , preferably satisfying $0 \leq z < p$. More precisely, any `t_INT` can be used as an **Fp**, but reduced inputs are treated more efficiently. Outputs from `Fpxxx` routines are reduced.

Fq: $\mathbf{Z}[X]/(p, T(X))$, p a `t_INT`, T a `t_POL` with **Fp** coefficients or `NULL` (in which case no reduction modulo T is performed). Implemented as `t_POLs` z with **Fp** coefficients, $\deg(z) < \deg T$, although z a `t_INT` is allowed for elements in the prime field.

Z: the integers \mathbf{Z} , implemented as `t_INTs`.

Zp: the p -adic integers \mathbf{Z}_p , implemented as `t_INTs`, for arbitrary p

Z1: the p -adic integers \mathbf{Z}_p , implemented as `t_INTs`, for $p < 2^{\text{BITS_IN_LONG}}$

z: the integers \mathbf{Z} , implemented using (signed) `longs`.

Q: the rational numbers \mathbf{Q} , implemented as `t_INTs` and `t_FRACs`.

Rg: a commutative ring, whose elements can be `gadd`-ed, `gmul`-ed, etc.

Possible letters are:

X: polynomial in X (`t_POL` in a fixed variable), e.g. `FpX` means $\mathbf{Z}/p\mathbf{Z}[X]$

Y: polynomial in $Y \neq X$. This is used to resolve ambiguities. E.g. `FpXY` means $((\mathbf{Z}/p\mathbf{Z})[X])[Y]$.

V: vector (`t_VEC` or `t_COL`), treated as a line vector (independently of the actual type). E.g. `ZV` means \mathbf{Z}^k for some k .

C: vector (`t_VEC` or `t_COL`), treated as a column vector (independently of the actual type). The difference with **V** is purely semantic: if the result is a vector, it will be of type `t_COL` unless mentioned otherwise. For instance the function `ZC_add` receives two integral vectors (`t_COL` or `t_VEC`, possibly different types) of the same length and returns a `t_COL` whose entries are the sums of the input coefficients.

M: matrix (`t_MAT`). E.g. `QM` means a matrix with rational entries

T: Trees. Either a leaf or a `t_VEC` of trees.

E: point over an elliptic curve, represented as two-component vectors `[x,y]`, except for the represented by the one-component vector `[0]`. Not all curve models are supported.

Q: representative (**t_POL**) of a class in a polynomial quotient ring. E.g. an **FpXQ** belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(T(X))$, **FpXQV** means a vector of such elements, etc.

n: a polynomial representative (**t_POL**) for a truncated power series modulo X^n . E.g. an **FpXn** belongs to $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$, **FpXnV** means a vector of such elements, etc.

x, y, m, v, c, q: as their uppercase counterpart, but coefficient arrays are implemented using **t_VECSMALLs**, which coefficient understood as **ulongs**.

x and **y** (and **q**) are implemented by a **t_VECSMALL** whose first coefficient is used as a code-word and the following are the coefficients, similarly to a **t_POL**. This is known as a 'POLSMALL'.

m are implemented by a **t_MAT** whose components (columns) are **t_VECSMALLs**. This is known as a 'MATSMALL'.

v and **c** are regular **t_VECSMALLs**. Difference between the two is purely semantic.

Omitting the letter means the argument is a scalar in the base ring. Standard functions *fun* are

add: add

sub: subtract

mul: multiply

sqr: square

div: divide (Euclidean quotient)

rem: Euclidean remainder

divrem: return Euclidean quotient, store remainder in a pointer argument. Three special values of that pointer argument modify the default behavior: **NULL** (do not store the remainder, used to implement **div**), **ONLY_REM** (return the remainder, used to implement **rem**), **ONLY_DIVIDES** (return the quotient if the division is exact, and **NULL** otherwise).

gcd: GCD

extgcd: return GCD, store Bezout coefficients in pointer arguments

pow: exponentiate

eval: evaluation / composition

7.2 Coefficient ring.

`long Rg_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the object x is defined.

Raise an error if it detects consistency problems in modular objects: incompatible rings (e.g. \mathbf{F}_p and \mathbf{F}_q for primes $p \neq q$, $\mathbf{F}_p[X]/(T)$ and $\mathbf{F}_p[X]/(U)$ for $T \neq U$). Minor discrepancies are supported if they make general sense (e.g. \mathbf{F}_p and \mathbf{F}_{p^k} , but not \mathbf{F}_p and \mathbf{Q}_p); `t_FFELT` and `t_POLMOD` of `t_INTMODs` are considered inconsistent, even if they define the same field: if you need to use simultaneously these different finite field implementations, multiply the polynomial by a `t_FFELT` equal to 1 first.

- 0: none of the others (presumably multivariate, possibly inconsistent).
- `t_INT`: defined over \mathbf{Z} .
- `t_FRAC`: defined over \mathbf{Q} .
- `t_INTMOD`: defined over $\mathbf{Z}/p\mathbf{Z}$, where `*ptp` is set to p . It is not checked whether p is prime.
- `t_COMPLEX`: defined over \mathbf{C} (at least one `t_COMPLEX` with at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_REAL`: defined over \mathbf{R} (at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_PADIC`: defined over \mathbf{Q}_p , where `*ptp` is set to p and `*ptprec` to the p -adic accuracy.
- `t_FFELT`: defined over a finite field \mathbf{F}_{p^k} , where `*ptp` is set to the field characteristic p and `*ptpol` is set to a `t_FFELT` belonging to the field.
- `t_POL`: defined over a polynomial ring.
- other values are composite corresponding to quotients $R[X]/(T)$, with one primary type `t1`, describing the form of the quotient, and a secondary type `t2`, describing R . If `t` is the `RgX_type`, `t1` and `t2` are recovered using

```
void RgX_type_decode(long t, long *t1, long *t2)
```

`t1` is one of

`t_POLMOD`: at least one `t_POLMOD` component, set `*ppol` to the modulus,

`t_QUAD`: no `t_POLMOD`, at least one `t_QUAD` component, set `*ppol` to the modulus (`-.pol`) of the `t_QUAD`,

`t_COMPLEX`: no `t_POLMOD` or `t_QUAD`, at least one `t_COMPLEX` component, set `*ppol` to $y^2 + 1$.

and the underlying base ring R is given by `t2`, which is one of `t_INT`, `t_INTMOD` (set `*ptp`) or `t_PADIC` (set `*ptp` and `*ptprec`), with the same meaning as above.

`int RgX_type_is_composite(long t)` t as returned by `RgX_type`, return 1 if t is a composite type, and 0 otherwise.

`GEN Rg_get_0(GEN x)` returns 0 in the base ring over which x is defined, to the proper accuracy (e.g. 0, Mod(0,3), 0(5¹⁰)).

`GEN Rg_get_1(GEN x)` returns 1 in the base ring over which x is defined, to the proper accuracy (e.g. 0, Mod(0,3),

`long RgX_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomial x is defined, otherwise as `Rg_type`.

`long RgX_Rg_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomial x and the element y are defined, otherwise as `Rg_type`.

`long RgX_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomials x and y are defined, otherwise as `Rg_type`.

`long RgX_type3(GEN x, GEN y, GEN z, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomials x , y and z are defined, otherwise as `Rg_type`.

`long RgM_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrix x is defined, otherwise as `Rg_type`.

`long RgM_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrices x and y are defined, otherwise as `Rg_type`.

`long RgM_RgC_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrix x and the vector y are defined, otherwise as `Rg_type`.

7.3 Modular arithmetic.

These routines implement univariate polynomial arithmetic and linear algebra over finite fields, in fact over finite rings of the form $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, where p is not necessarily prime and $T \in (\mathbf{Z}/p\mathbf{Z})[X]$ is possibly reducible; and finite extensions thereof. All this can be emulated with `t_INTMOD` and `t_POLMOD` coefficients and using generic routines, at a considerable loss of efficiency. Also, specialized routines are available that have no obvious generic equivalent.

7.3.1 FpC / FpV, FpM. A ZV (resp. a ZM) is a `t_VEC` or `t_COL` (resp. `t_MAT`) with `t_INT` coefficients. An FpV or FpM , with respect to a given `t_INT` p , is the same with Fp coordinates; operations are understood over $\mathbf{Z}/p\mathbf{Z}$.

7.3.1.1 Conversions.

`int Rg_is_Fp(GEN z, GEN *p)`, checks if z can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT` or a `t_INTMOD` whose modulus is equal to $*p$, (if $*p$ not `NULL`), in that case return 1, else 0. If a modulus is found it is put in $*p$, else $*p$ is left unchanged.

`int RgV_is_FpV(GEN z, GEN *p)`, z a `t_VEC` (resp. `t_COL`), checks if it can be mapped to a FpV (resp. FpC), by checking `Rg_is_Fp` coefficientwise.

`int RgM_is_FpM(GEN z, GEN *p)`, z a `t_MAT`, checks if it can be mapped to a FpM , by checking `RgV_is_FpV` columnwise.

`GEN Rg_to_Fp(GEN z, GEN p)`, z a scalar which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a `t_INT`, a `t_INTMOD` whose modulus is divisible by p , a `t_FRAC` whose denominator is coprime to p , or a `t_PADIC` with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized.

`GEN padic_to_Fp(GEN x, GEN p)` special case of `Rg_to_Fp`, for a x a `t_PADIC`.

`GEN RgV_to_FpV(GEN z, GEN p)`, z a `t_VEC` or `t_COL`, returns the FpV (as a `t_VEC`) obtained by applying `Rg_to_Fp` coefficientwise.

GEN RgC_to_FpC(GEN z, GEN p), z a `t_VEC` or `t_COL`, returns the FpC (as a `t_COL`) obtained by applying `Rg_to_Fp` coefficientwise.

GEN RgM_to_FpM(GEN z, GEN p), z a `t_MAT`, returns the FpM obtained by applying `RgC_to_FpC` columnwise.

GEN RgM_Fp_init(GEN z, GEN p, ulong *pp), given an RgM z , whose entries can be mapped to \mathbf{F}_p (as per `Rg_to_Fp`), and a prime number p . This routine returns a normal form of z : either an F2m ($p = 2$), an Flm (p fits into an `ulong`) or an FpM. In the first two cases, `pp` is set to `itou(p)`, and to 0 in the last.

The functions above are generally used as follow:

```
GEN add(GEN x, GEN y)
{
  GEN p = NULL;
  if (Rg_is_Fp(x, &p) && Rg_is_Fp(y, &p) && p)
  {
    x = Rg_to_Fp(x, p); y = Rg_to_Fp(y, p);
    z = Fp_add(x, y, p);
    return Fp_to_mod(z);
  }
  else return gadd(x, y);
}
```

GEN FpC_red(GEN z, GEN p), z a ZC. Returns `lift(Col(z) * Mod(1,p))`, hence a `t_COL`.

GEN FpV_red(GEN z, GEN p), z a ZV. Returns `lift(Vec(z) * Mod(1,p))`, hence a `t_VEC`

GEN FpM_red(GEN z, GEN p), z a ZM. Returns `lift(z * Mod(1,p))`, which is an FpM.

7.3.1.2 Basic operations.

GEN random_FpC(long n, GEN p) returns a random FpC with n components.

GEN random_FpV(long n, GEN p) returns a random FpV with n components.

GEN FpC_center(GEN z, GEN p, GEN pov2) returns a `t_COL` whose entries are the `Fp_center` of the `gel(z,i)`.

GEN FpM_center(GEN z, GEN p, GEN pov2) returns a matrix whose entries are the `Fp_center` of the `gcoeff(z,i,j)`.

void FpC_center_inplace(GEN z, GEN p, GEN pov2) in-place version of `FpC_center`, using `affii`.

void FpM_center_inplace(GEN z, GEN p, GEN pov2) in-place version of `FpM_center`, using `affii`.

GEN FpC_add(GEN x, GEN y, GEN p) adds the ZC x and y and reduce modulo p to obtain an FpC.

GEN FpV_add(GEN x, GEN y, GEN p) same as `FpC_add`, returning and FpV.

GEN FpM_add(GEN x, GEN y, GEN p) adds the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpC_sub(GEN x, GEN y, GEN p) subtracts the ZC y to the ZC x and reduce modulo p to obtain an FpC.

GEN FpV_sub(GEN x, GEN y, GEN p) same as FpC_sub, returning and FpV.

GEN FpM_sub(GEN x, GEN y, GEN p) subtracts the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpC_Fp_mul(GEN x, GEN y, GEN p) multiplies the ZC x (seen as a column vector) by the t_INT y and reduce modulo p to obtain an FpC.

GEN FpM_Fp_mul(GEN x, GEN y, GEN p) multiplies the ZM x (seen as a column vector) by the t_INT y and reduce modulo p to obtain an FpM.

GEN FpC_FpV_mul(GEN x, GEN y, GEN p) multiplies the ZC x (seen as a column vector) by the ZV y (seen as a row vector, assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpM_mul(GEN x, GEN y, GEN p) multiplies the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpM_powu(GEN x, ulong n, GEN p) computes x^n where x is a square FpM.

GEN FpM_FpC_mul(GEN x, GEN y, GEN p) multiplies the ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions), and reduce modulo p to obtain an FpC.

GEN FpM_FpC_mul_FpX(GEN x, GEN y, GEN p, long v) is a memory-clean version of

```
GEN tmp = FpM_FpC_mul(x,y,p);
return RgV_to_RgX(tmp, v);
```

GEN FpV_FpC_mul(GEN x, GEN y, GEN p) multiplies the ZV x (seen as a row vector) by the ZC y (seen as a column vector, assumed to have compatible dimensions), and reduce modulo p to obtain an Fp.

GEN FpV_dotproduct(GEN x, GEN y, GEN p) scalar product of x and y (assumed to have the same length).

GEN FpV_dotsquare(GEN x, GEN p) scalar product of x with itself. has t_INT entries.

GEN FpV_factorback(GEN L, GEN e, GEN p) given an FpV L and a ZV e of the same length, return $\prod_i L_i^{e_i}$ modulo p.

7.3.1.3 Fp-linear algebra. The implementations are not asymptotically efficient ($O(n^3)$ standard algorithms).

GEN FpM_deplin(GEN x, GEN p) returns a non-trivial kernel vector, or NULL if none exist.

GEN FpM_det(GEN x, GEN p) as det

GEN FpM_gauss(GEN a, GEN b, GEN p) as gauss, where a and b are FpM.

GEN FpM_FpC_gauss(GEN a, GEN b, GEN p) as gauss, where a is a FpM and b a FpC.

GEN FpM_image(GEN x, GEN p) as image

GEN FpM_intersect(GEN x, GEN y, GEN p) as intersect

GEN FpM_inv(GEN x, GEN p) returns a left inverse of x (the inverse if x is square), or NULL if x is not invertible.

GEN FpM_FpC_invimage(GEN A, GEN y, GEN p) given an FpM A and an FpC y, returns an x such that $Ax = y$, or NULL if no such vector exist.

GEN FpM_invimage(GEN A, GEN y, GEN p) given two FpM A and y , returns x such that $Ax = y$, or NULL if no such matrix exist.

GEN FpM_ker(GEN x, GEN p) as ker

long FpM_rank(GEN x, GEN p) as rank

GEN FpM_indexrank(GEN x, GEN p) as indexrank

GEN FpM_suppl(GEN x, GEN p) as suppl

GEN FpM_hess(GEN x, GEN p) upper Hessenberg form of x over \mathbf{F}_p .

GEN FpM_charpoly(GEN x, GEN p) characteristic polynomial of x .

7.3.1.4 FqC, FqM and Fq-linear algebra.

An FqM (resp. FqC) is a matrix (resp a t_COL) with Fq coefficients (with respect to given T, p), not necessarily reduced (i.e arbitrary t_INTs and ZXs in the same variable as T).

GEN RgC_to_FqC(GEN z, GEN T, GEN p)

GEN RgM_to_FqM(GEN z, GEN T, GEN p)

GEN FqC_add(GEN a, GEN b, GEN T, GEN p)

GEN FqC_sub(GEN a, GEN b, GEN T, GEN p)

GEN FqC_Fq_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM_FqC_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where b is a FqC.

GEN FqM_FqC_invimage(GEN a, GEN b, GEN T, GEN p)

GEN FqM_FqC_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM_deplin(GEN x, GEN T, GEN p) returns a non-trivial kernel vector, or NULL if none exist.

GEN FqM_det(GEN x, GEN T, GEN p) as det

GEN FqM_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where b is a FqM.

GEN FqM_image(GEN x, GEN T, GEN p) as image

GEN FqM_indexrank(GEN x, GEN T, GEN p) as indexrank

GEN FqM_inv(GEN x, GEN T, GEN p) returns the inverse of x , or NULL if x is not invertible.

GEN FqM_invimage(GEN a, GEN b, GEN T, GEN p) as invimage

GEN FqM_ker(GEN x, GEN T, GEN p) as ker

GEN FqM_mul(GEN a, GEN b, GEN T, GEN p)

long FqM_rank(GEN x, GEN T, GEN p) as rank

GEN FqM_suppl(GEN x, GEN T, GEN p) as suppl

7.3.2 Flc / Flv, Flm. See FpV, FpM operations.

GEN Flv_copy(GEN x) returns a copy of x.

GEN Flv_center(GEN z, ulong p, ulong ps2)

GEN random_Flv(long n, ulong p) returns a random Flv with n components.

GEN Flm_copy(GEN x) returns a copy of x.

GEN matid_Flm(long n) returns an Flm which is an $n \times n$ identity matrix.

GEN scalar_Flm(long s, long n) returns an Flm which is s times the $n \times n$ identity matrix.

GEN Flm_center(GEN z, ulong p, ulong ps2)

GEN Flm_Fl_add(GEN x, ulong y, ulong p) returns $x + y * \text{Id}$ (x must be square).

GEN Flm_Flc_mul(GEN x, GEN y, ulong p) multiplies x and y (assumed to have compatible dimensions).

GEN Flm_Flc_mul_pre(GEN x, GEN y, ulong p, ulong pi) multiplies x and y (assumed to have compatible dimensions), assuming pi is the pseudo inverse of p .

GEN Flc_Flv_mul(GEN x, GEN y, ulong p) multiplies the column vector x by the row vector y . The result is a matrix.

GEN Flm_Flc_mul_pre_Flx(GEN x, GEN y, ulong p, ulong pi, long sv) return Flv_to_Flx(Flm_Flc_mul_pre(x, y, p, pi), sv).

GEN Flm_Fl_mul(GEN x, ulong y, ulong p) multiplies the Flm x by y .

GEN Flm_neg(GEN x, ulong p) negates the Flm x .

void Flm_Fl_mul_inplace(GEN x, ulong y, ulong p) replaces the Flm x by $x * y$.

GEN Flv_Fl_mul(GEN x, ulong y, ulong p) multiplies the Flv x by y .

void Flv_Fl_mul_inplace(GEN x, ulong y, ulong p) replaces the Flc x by $x * y$.

void Flv_Fl_mul_part_inplace(GEN x, ulong y, ulong p, long l) multiplies $x[1..l]$ by y modulo p . In place.

GEN Flv_Fl_div(GEN x, ulong y, ulong p) divides the Flv x by y .

void Flv_Fl_div_inplace(GEN x, ulong y, ulong p) replaces the Flv x by x/y .

void Flc_lincomb1_inplace(GEN X, GEN Y, ulong v, ulong q) sets $X \leftarrow X + vY$, where X, Y are Flc. Memory efficient (e.g. no-op if $v = 0$), and gerepile-safe.

GEN Flv_add(GEN x, GEN y, ulong p) adds two Flv.

void Flv_add_inplace(GEN x, GEN y, ulong p) replaces x by $x + y$.

GEN Flv_neg(GEN x, ulong p) returns $-x$.

void Flv_neg_inplace(GEN x, ulong p) replaces x by $-x$.

GEN Flv_sub(GEN x, GEN y, ulong p) subtracts y to x .

void Flv_sub_inplace(GEN x, GEN y, ulong p) replaces x by $x - y$.

ulong Flv_dotproduct(GEN x, GEN y, ulong p) returns the scalar product of x and y

`ulong Flv_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)` returns the scalar product of x and y assuming pi is the pseudo inverse of p .

`ulong Flv_sum(GEN x, ulong p)` returns the sum of the components of x .

`ulong Flv_prod(GEN x, ulong p)` returns the product of the components of x .

`ulong Flv_prod_pre(GEN x, ulong p, ulong pi)` as `Flv_prod` assuming pi is the pseudo inverse of p .

`GEN Flv_inv(GEN x, ulong p)` returns the vector of inverses of the elements of x (as a `Flv`). Use Montgomery trick.

`void Flv_inv_inplace(GEN x, ulong p)` in place variant of `Flv_inv`.

`GEN Flv_inv_pre(GEN x, ulong p, ulong pi)` as `Flv_inv` assuming pi is the pseudo inverse of p .

`void Flv_inv_pre_inplace(GEN x, ulong p, ulong pi)` in place variant of `Flv_inv`.

`GEN Flc_FpV_mul(GEN x, GEN y, GEN p)` multiplies x (seen as a column vector) by y (seen as a row vector, assumed to have compatible dimensions) to obtain an `Flm`.

`GEN zero_Flm(long m, long n)` creates a `Flm` with $m \times n$ components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_Flm_copy(long m, long n)` creates a `Flm` with $m \times n$ components set to 0.

`GEN zero_Flv(long n)` creates a `Flv` with n components set to 0.

`GEN Flm_row(GEN A, long x0)` return $A[i,]$, the i -th row of the `Flm` A .

`GEN Flm_add(GEN x, GEN y, ulong p)` adds x and y (assumed to have compatible dimensions).

`GEN Flm_sub(GEN x, GEN y, ulong p)` subtracts x and y (assumed to have compatible dimensions).

`GEN Flm_mul(GEN x, GEN y, ulong p)` multiplies x and y (assumed to have compatible dimensions).

`GEN Flm_powers(GEN x, ulong n, ulong p)` returns $[x^0, \dots, x^n]$ as a `t_VEC` of `Flms`.

`GEN Flm_powu(GEN x, ulong n, ulong p)` computes x^n where x is a square `Flm`.

`GEN Flm_charpoly(GEN x, ulong p)` return the characteristic polynomial of the square `Flm` x , as a `Flx`.

`GEN Flm_deplin(GEN x, ulong p)`

`ulong Flm_det(GEN x, ulong p)`

`ulong Flm_det_sp(GEN x, ulong p)`, as `Flm_det`, in place (destroys x).

`GEN Flm_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flm`.

`GEN Flm_Flc_gauss(GEN a, GEN b, ulong p)` as `gauss`, where b is a `Flc`.

`GEN Flm_indexrank(GEN x, ulong p)`

`GEN Flm_inv(GEN x, ulong p)`

`GEN Flm_adjoint(GEN x, ulong p)` as `matadjoint`.

GEN Flm_Flc_invimage(GEN A, GEN y, ulong p) given an Flm A and an Flc y , returns an x such that $Ax = y$, or NULL if no such vector exist.

GEN Flm_invimage(GEN A, GEN y, ulong p) given two Flm A and y , returns x such that $Ax = y$, or NULL if no such matrix exist.

GEN Flm_ker(GEN x, ulong p)

GEN Flm_ker_sp(GEN x, ulong p, long deplin), as Flm_ker (if deplin=0) or Flm.deplin (if deplin=1), in place (destroys x).

long Flm_rank(GEN x, ulong p)

long Flm_suppl(GEN x, ulong p)

GEN Flm_image(GEN x, ulong p)

GEN Flm_intersect(GEN x, GEN y, ulong p)

GEN Flm_transpose(GEN x)

GEN Flm_hess(GEN x, ulong p) upper Hessenberg form of x over \mathbf{F}_p .

7.3.3 F2c / F2v, F2m. An F2v v is a `t_VECSMALL` representing a vector over \mathbf{F}_2 . Specifically $z[0]$ is the usual codeword, $z[1]$ is the number of components of v and the coefficients are given by the bits of remaining words by increasing indices.

ulong F2v_coeff(GEN x, long i) returns the coefficient $i \geq 1$ of x .

void F2v_clear(GEN x, long i) sets the coefficient $i \geq 1$ of x to 0.

void F2v_flip(GEN x, long i) adds 1 to the coefficient $i \geq 1$ of x .

void F2v_set(GEN x, long i) sets the coefficient $i \geq 1$ of x to 1.

void F2v_copy(GEN x) returns a copy of x .

GEN F2v_slice(GEN x, long a, long b) returns the F2v with entries $x[a], \dots, x[b]$. Assumes $a \leq b$.

ulong F2m_coeff(GEN x, long i, long j) returns the coefficient (i, j) of x .

void F2m_clear(GEN x, long i, long j) sets the coefficient (i, j) of x to 0.

void F2m_flip(GEN x, long i, long j) adds 1 to the coefficient (i, j) of x .

void F2m_set(GEN x, long i, long j) sets the coefficient (i, j) of x to 1.

void F2m_copy(GEN x) returns a copy of x .

GEN F2m_rowslice(GEN x, long a, long b) returns the F2m built from the a -th to b -th rows of the F2m x . Assumes $a \leq b$.

GEN F2m_F2c_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN F2m_image(GEN x) gives a subset of the columns of x that generate the image of x .

GEN F2m_invimage(GEN A, GEN B)

GEN F2m_F2c_invimage(GEN A, GEN y)

GEN F2m_gauss(GEN a, GEN b) as `gauss`, where b is a F2m.

GEN F2m_F2c_gauss(GEN a, GEN b) as gauss, where b is a F2c.

GEN F2m_indexrank(GEN x) x being a matrix of rank r , returns a vector with two `t_VECSMALL` components y and z of length r giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using `vecextract(x, y, z)` is invertible.

GEN F2m_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN F2m_powu(GEN x, ulong n) computes x^n where x is a square F2m.

long F2m_rank(GEN x) as rank.

long F2m_suppl(GEN x) as suppl.

GEN matid_F2m(long n) returns an F2m which is an $n \times n$ identity matrix.

GEN zero_F2v(long n) creates a F2v with n components set to 0.

GEN const_F2v(long n) creates a F2v with n components set to 1.

GEN F2v_ei(long n, long i) creates a F2v with n components set to 0, but for the i -th one, which is set to 1 (i -th vector in the canonical basis).

GEN zero_F2m(long m, long n) creates a F2m with $m \times n$ components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

GEN zero_F2m_copy(long m, long n) creates a F2m with $m \times n$ components set to 0.

GEN F2v_to_Flv(GEN x)

GEN F2c_to_ZC(GEN x)

GEN ZV_to_F2v(GEN x)

GEN RgV_to_F2v(GEN x)

GEN F2m_to_Flm(GEN x)

GEN F2m_to_ZM(GEN x)

GEN Flv_to_F2v(GEN x)

GEN Flm_to_F2m(GEN x)

GEN ZM_to_F2m(GEN x)

GEN RgM_to_F2m(GEN x)

void F2v_add_inplace(GEN x, GEN y) replaces x by $x + y$. It is allowed for y to be shorter than x .

ulong F2m_det(GEN x)

ulong F2m_det_sp(GEN x), as F2m_det, in place (destroys x).

GEN F2m_deplin(GEN x)

ulong F2v_dotproduct(GEN x, GEN y) returns the scalar product of x and y

GEN F2m_inv(GEN x)

GEN F2m_ker(GEN x)

GEN F2m_ker_sp(GEN x, long deplin), as F2m_ker (if `deplin=0`) or F2m_deplin (if `deplin=1`), in place (destroys x).

7.3.4 FlxqV, FlxqC, FlxqM. See FqV, FqC, FqM operations.

GEN FlxqV_dotproduct(GEN x, GEN y, GEN T, ulong p) as FpV_dotproduct.
 GEN FlxM_Flx_add_shallow(GEN x, GEN y, ulong p) as RgM_Rg_add_shallow.
 GEN FlxqC_Flxq_mul(GEN x, GEN y, GEN T, ulong p)
 GEN FlxqM_Flxq_mul(GEN x, GEN y, GEN T, ulong p)
 GEN FlxqM_FlxqC_gauss(GEN a, GEN b, GEN T, ulong p)
 GEN FlxqM_FlxqC_invimage(GEN a, GEN b, GEN T, ulong p)
 GEN FlxqM_FlxqC_mul(GEN a, GEN b, GEN T, ulong p)
 GEN FlxqM_deplin(GEN x, GEN T, ulong p)
 GEN FlxqM_det(GEN x, GEN T, ulong p)
 GEN FlxqM_gauss(GEN a, GEN b, GEN T, ulong p)
 GEN FlxqM_image(GEN x, GEN T, ulong p)
 GEN FlxqM_indexrank(GEN x, GEN T, ulong p)
 GEN FlxqM_inv(GEN x, GEN T, ulong p)
 GEN FlxqM_invimage(GEN a, GEN b, GEN T, ulong p)
 GEN FlxqM_ker(GEN x, GEN T, ulong p)
 GEN FlxqM_mul(GEN a, GEN b, GEN T, ulong p)
 long FlxqM_rank(GEN x, GEN T, ulong p)
 GEN FlxqM_suppl(GEN x, GEN T, ulong p)
 GEN matid_FlxqM(long n, GEN T, ulong p)

7.3.5 FpX. Let p an understood t_INT , to be given in the function arguments; in practice p is not assumed to be prime, but be wary. Recall that an Fp object is a t_INT , preferably belonging to $[0, p - 1]$; an FpX is a t_POL in a fixed variable whose coefficients are Fp objects. Unless mentioned otherwise, all outputs in this section are FpXs. All operations are understood to take place in $(\mathbf{Z}/p\mathbf{Z})[X]$.

7.3.5.1 Conversions. In what follows p is always a t_INT , not necessarily prime.

int RgX_is_FpX(GEN z, GEN *p), z a t_POL , checks if it can be mapped to a FpX, by checking Rg_is_Fp coefficientwise.

GEN RgX_to_FpX(GEN z, GEN p), z a t_POL , returns the FpX obtained by applying Rg_to_Fp coefficientwise.

GEN FpX_red(GEN z, GEN p), z a ZX, returns $\text{lift}(z * \text{Mod}(1, p))$, normalized.

GEN FpXV_red(GEN z, GEN p), z a t_VEC of ZX. Applies FpX_red componentwise and returns the result (and we obtain a vector of FpXs).

GEN FpXT_red(GEN z, GEN p), z a tree of ZX. Applies FpX_red to each leaf and returns the result (and we obtain a tree of FpXs).

7.3.5.2 Basic operations. In what follows p is always a $\mathfrak{t_INT}$, not necessarily prime.

Now, except for p , the operands and outputs are all \mathfrak{FpX} objects. Results are undefined on other inputs.

$\text{GEN FpX_add}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ adds x and y .

$\text{GEN FpX_neg}(\text{GEN } x, \text{GEN } p)$ returns $-x$, the components are between 0 and p if this is the case for the components of x .

$\text{GEN FpX_renormalize}(\text{GEN } x, \text{long } l)$, as `normalizpol`, where $l = \lg(x)$, in place.

$\text{GEN FpX_sub}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ returns $x - y$.

$\text{GEN FpX_halve}(\text{GEN } x, \text{GEN } p)$ returns z such that $2z = x$ modulo p assuming such z exists.

$\text{GEN FpX_mul}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ returns xy .

$\text{GEN FpX_mulspec}(\text{GEN } a, \text{GEN } b, \text{GEN } p, \text{long } na, \text{long } nb)$ see `ZX_mulspec`

$\text{GEN FpX_sqr}(\text{GEN } x, \text{GEN } p)$ returns x^2 .

$\text{GEN FpX_powu}(\text{GEN } x, \text{ulong } n, \text{GEN } p)$ returns x^n .

$\text{GEN FpX_convol}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ return the-term by-term product of x and y .

$\text{GEN FpX_divrem}(\text{GEN } x, \text{GEN } y, \text{GEN } p, \text{GEN } *pr)$ returns the quotient of x by y , and sets pr to the remainder.

$\text{GEN FpX_div}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ returns the quotient of x by y .

$\text{GEN FpX_div_by_X_x}(\text{GEN } A, \text{GEN } a, \text{GEN } p, \text{GEN } *r)$ returns the quotient of the \mathfrak{FpX} A by $(X - a)$, and sets r to the remainder $A(a)$.

$\text{GEN FpX_rem}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ returns the remainder $x \bmod y$.

$\text{long FpX_valrem}(\text{GEN } x, \text{GEN } t, \text{GEN } p, \text{GEN } *r)$ The arguments x and e being non-zero \mathfrak{FpX} returns the highest exponent e such that \mathfrak{t}^e divides x . The quotient x/\mathfrak{t}^e is returned in $*r$. In particular, if t is irreducible, this returns the valuation at t of x , and $*r$ is the prime-to- t part of x .

$\text{GEN FpX_deriv}(\text{GEN } x, \text{GEN } p)$ returns the derivative of x . This function is not memory-clean, but nevertheless suitable for `gerepileupto`.

$\text{GEN FpX_integ}(\text{GEN } x, \text{GEN } p)$ returns the primitive of x whose constant term is 0.

$\text{GEN FpX_digits}(\text{GEN } x, \text{GEN } B, \text{GEN } p)$ returns a vector of \mathfrak{FpX} $[c_0, \dots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^n c_i B^i$.

$\text{GEN FpXV_FpX_fromdigits}(\text{GEN } v, \text{GEN } B, \text{GEN } p)$ where $v = [c_0, \dots, c_n]$ is a vector of \mathfrak{FpX} , returns $\sum_{i=0}^n c_i B^i$.

$\text{GEN FpX_translate}(\text{GEN } P, \text{GEN } c, \text{GEN } p)$ let c be an \mathfrak{Fp} and let P be an \mathfrak{FpX} ; returns the translated \mathfrak{FpX} of $P(X + c)$.

$\text{GEN FpX_gcd}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ returns a (not necessarily monic) greatest common divisor of x and y .

$\text{GEN FpX_halfgcd}(\text{GEN } x, \text{GEN } y, \text{GEN } p)$ returns a two-by-two \mathfrak{FpXM} M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN FpX_extgcd(GEN x, GEN y, GEN p, GEN *u, GEN *v) returns $d = \text{GCD}(x, y)$ (not necessarily monic), and sets *u, *v to the Bezout coefficients such that $*ux + *vy = d$. If *u is set to NULL, it is not computed which is a bit faster. This is useful when computing the inverse of y modulo x .

GEN FpX_center(GEN z, GEN p, GEN pov2) returns the polynomial whose coefficient belong to the symmetric residue system. Assumes the coefficients already belong to $] -p/2, p[$ and that pov2 is shifti(p, -1).

GEN FpX_center_i(GEN z, GEN p, GEN pov2) internal variant of FpX_center, not gerepile-safe.

GEN FpX_Frobenius(GEN T, GEN p) returns $X^p \pmod{T(X)}$.

GEN FpX_matFrobenius(GEN T, GEN p) returns the matrix of the Frobenius automorphism $x \mapsto x^p$ over the power basis of $\mathbf{F}_p[X]/(T)$.

7.3.5.3 Mixed operations. The following functions implement arithmetic operations between FpX and Fp operands, the result being of type FpX. The integer p need not be prime.

GEN Z_to_FpX(GEN x, GEN p, long v) converts a t_INT to a scalar polynomial in variable v , reduced modulo p .

GEN FpX_Fp_add(GEN y, GEN x, GEN p) add the Fp x to the FpX y.

GEN FpX_Fp_add_shallow(GEN y, GEN x, GEN p) add the Fp x to the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN FpX_Fp_sub(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y.

GEN FpX_Fp_sub_shallow(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN Fp_FpX_sub(GEN x, GEN y, GEN p) returns $x - y$, where x is a t_INT and y an FpX.

GEN FpX_Fp_mul(GEN x, GEN y, GEN p) multiplies the FpX x by the Fp y.

GEN FpX_Fp_mulspec(GEN x, GEN y, GEN p, long lx) see ZX_mulspec

GEN FpX_mulu(GEN x, ulong y, GEN p) multiplies the FpX x by y.

GEN FpX_Fp_mul_to_monic(GEN y, GEN x, GEN p) returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

7.3.5.4 Miscellaneous operations.

GEN FpX_normalize(GEN z, GEN p) divides the FpX z by its leading coefficient. If the latter is 1, z itself is returned, not a copy. If not, the inverse remains uncollected on the stack.

GEN FpX_invBarrett(GEN T, GEN p), returns the Barrett inverse M of T defined by $M(x)x^n \times T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T .

GEN FpX_rescale(GEN P, GEN h, GEN p) returns $h^{\deg(P)}P(x/h)$. P is an FpX and h is a non-zero Fp (the routine would work with any non-zero t_INT but is not efficient in this case).

GEN FpX_eval(GEN x, GEN y, GEN p) evaluates the FpX x at the Fp y. The result is an Fp.

GEN FpX_FpV_multieval(GEN P, GEN v, GEN p) returns the vector $[P(v[1]), \dots, P(v[n])]$ as a FpV.

GEN FpX_dotproduct(GEN x, GEN y, GEN p) return the scalar product $\sum_{i \geq 0} x_i y_i$ of the coefficients of x and y .

GEN FpXV_FpC_mul(GEN V, GEN W, GEN p) multiplies a non-empty line vector of FpX by a column vector of Fp of compatible dimensions. The result is an FpX.

GEN FpXV_prod(GEN V, GEN p), V being a vector of FpX, returns their product.

GEN FpV_roots_to_pol(GEN V, GEN p, long v), V being a vector of INTs, returns the monic FpX $\prod_i (\text{pol_x}[v] - V[i])$.

GEN FpX_chinese_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz, GEN p): returns an FpX, congruent to x mod Tx and to y mod Ty. Assumes Tx and Ty are coprime, and Tz = Tx * Ty or NULL (in which case it is computed within).

GEN FpV_polint(GEN x, GEN y, GEN p, long v) returns the FpX interpolation polynomial with value y[i] at x[i]. Assumes lengths are the same, components are t_INTs, and the x[i] are distinct modulo p.

GEN FpV_FpM_polint(GEN x, GEN V, GEN p, long v) equivalent (but faster) to applying FpV_polint(x,...) to all the elements of the vector V (thus, returns a FpXV).

GEN FpV_invVandermonde(GEN L, GEN d, GEN p) L being a FpV of length n, return the inverse M of the Vandermonde matrix attached to the elements of L, eventually multiplied by d if it is not NULL. If A is a FpV and B = MA, then the polynomial $P = \sum_{i=1}^n B[i]X^{i-1}$ verifies $P(L[i]) = dA[i]$ for $1 \leq i \leq n$.

int FpX_is_squarefree(GEN f, GEN p) returns 1 if the FpX f is squarefree, 0 otherwise.

int FpX_is_irred(GEN f, GEN p) returns 1 if the FpX f is irreducible, 0 otherwise. Assumes that p is prime. If f has few factors, FpX_nbfact(f,p) == 1 is much faster.

int FpX_is_totally_split(GEN f, GEN p) returns 1 if the FpX f splits into a product of distinct linear factors, 0 otherwise. Assumes that p is prime.

long FpX_ispower(GEN f, ulong k, GEN p, GEN *pt) return 1 if the FpX f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

GEN FpX_factor(GEN f, GEN p), factors the FpX f. Assumes that p is prime. The returned value v is a t_VEC with two components: v[1] is a vector of distinct irreducible (FpX) factors, and v[2] is a t_VECSMALL of corresponding exponents. The order of the factors is deterministic (the computation is not).

GEN FpX_factor_squarefree(GEN f, GEN p) returns the squarefree factorization of f modulo p. This is a vector $[u_1, \dots, u_k]$ of pairwise coprime FpX such that $u_k \neq 1$ and $f = \prod u_i^i$. Shallow function.

GEN FpX_ddf(GEN f, GEN p) assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value v is a t_VEC with two components: F=v[1] is a vector of (FpX) factors, and E=v[2] is a t_VECSMALL, such that f is equal to the product of the F[i] and each F[i] is a product of irreducible factors of degree E[i].

long FpX_ddf_degree(GEN f, GEN XP, GEN p) assuming that f is squarefree and that all its factors have the same degree, return the common degree, where XP is FpX_Frobenius(f, p).

long FpX_nbfact(GEN f, GEN p), assuming the FpX f is squarefree, returns the number of its irreducible factors. Assumes that p is prime.

long FpX_nbfact_Frobenius(GEN f, GEN XP, GEN p), as FpX_nbfact(f, p) but faster, where XP is FpX_Frobenius(f, p).

GEN FpX_degfact(GEN f, GEN p), as FpX_factor, but the degrees of the irreducible factors are returned instead of the factors themselves (as a t_VECSMALL). Assumes that p is prime.

long FpX_nbroots(GEN f, GEN p) returns the number of distinct roots in $\mathbf{Z}/p\mathbf{Z}$ of the FpX f. Assumes that p is prime.

GEN FpX_oneroot(GEN f, GEN p) returns one root in $\mathbf{Z}/p\mathbf{Z}$ of the FpX f. Return NULL if no root exists. Assumes that p is prime.

GEN FpX_oneroot_split(GEN f, GEN p) as FpX_oneroot. Faster when f is close to be totally split.

GEN FpX_roots(GEN f, GEN p) returns the roots in $\mathbf{Z}/p\mathbf{Z}$ of the FpX f (without multiplicity, as a vector of Fps). Assumes that p is prime.

GEN FpX_split_part(GEN f, GEN p) returns the largest totally split squarefree factor of f.

GEN random_FpX(long d, long v, GEN p) returns a random FpX in variable v, of degree less than d.

GEN FpX_resultant(GEN x, GEN y, GEN p) returns the resultant of x and y, both FpX. The result is a t_INT belonging to $[0, p-1]$.

GEN FpX_disc(GEN x, GEN p) returns the discriminant of the FpX x. The result is a t_INT belonging to $[0, p-1]$.

GEN FpX_FpXY_resultant(GEN a, GEN b, GEN p), a a t_POL of t_INTs (say in variable X), b a t_POL (say in variable X) whose coefficients are either t_POLs in $\mathbf{Z}[Y]$ or t_INTs. Returns $\text{Res}_X(a, b)$ in $\mathbf{F}_p[Y]$ as an FpY. The function assumes that X has lower priority than Y.

GEN FpX_Newton(GEN x, long n, GEN p) return $\sum i = 0^{n-1} \pi_i X^i$ where π_i is the sum of the *i*th-power of the roots of x in an algebraic closure.

GEN FpX_fromNewton(GEN x, GEN p) recover a polynomial from its Newton sums given by the coefficients of x. This function assumes that p and the accuracy of x as a FpXn is larger than the degree of the solution.

GEN FpX_Laplace(GEN x, GEN p) return $\sum i = 0^{n-1} x_i i! X^i$.

GEN FpX_invLaplace(GEN x, GEN p) return $\sum i = 0^{n-1} x_i / i! X^i$.

7.3.6 FpXQ, Fq. Let p a t_INT and T an FpX for p, both to be given in the function arguments; an FpXQ object is an FpX whose degree is strictly less than the degree of T. An Fq is either an FpXQ or an Fp. Both represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$, in which all operations below take place. In addition, Fq routines also allow T = NULL, in which case no reduction mod T is performed on the result.

For efficiency, the routines in this section may leave small unused objects behind on the stack (their output is still suitable for gerepileupto). Besides T and p, arguments are either FpXQ or Fq depending on the function name. (All Fq routines accept FpXQs by definition, not the other way round.)

7.3.6.1 Preconditioned reduction.

For faster reduction, the modulus T can be replaced by an extended modulus in all `FpXQ`- and `Fq`-classes functions, and in `FpX_rem` and `FpX_divrem`. An extended modulus (`FpXT`, which is a tree whose leaves are `FpX`) In current implementation, an extended modulus is either a plain modulus (an `FpX`) or a pair of polynomials, one being the plain modulus T and the other being `FpX_invBarret(T, p)`.

`GEN FpX_get_red(GEN T, GEN p)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FpX_mod(GEN eT)` returns the underlying modulus T .

`GEN get_FpX_var(GEN eT)` returns the variable number `varn(T)`.

`GEN get_FpX_degree(GEN eT)` returns the degree `degpol(T)`.

7.3.6.2 Conversions.

`GEN Rg_is_FpXQ(GEN z, GEN *T, GEN *p)`, checks if z is a `GEN` which can be mapped to $\mathbf{F}_p[X]/(T)$: anything for which `Rg_is_Fp` return 1, a `t_POL` for which `RgX_to_FpX` return 1, a `t_POLMOD` whose modulus is equal to $*T$ if $*T$ is not `NULL` (once mapped to a `FpX`), or a `t_FFELT` z with the same definition field as $*T$ if $*T$ is not `NULL` and is a `t_FFELT`.

If an integer modulus is found it is put in $*p$, else $*p$ is left unchanged. If a polynomial modulus is found it is put in $*T$, if a `t_FFELT` z is found, z is put in $*T$, else $*T$ is left unchanged.

`int RgX_is_FpXQX(GEN z, GEN *T, GEN *p)`, z a `t_POL`, checks if it can be mapped to a `FpXQX`, by checking `Rg_is_FpXQ` coefficientwise.

`GEN Rg_to_FpXQ(GEN z, GEN T, GEN p)`, z a `GEN` which can be mapped to $\mathbf{F}_p[X]/(T)$: anything `Rg_to_Fp` can be applied to, a `t_POL` to which `RgX_to_FpX` can be applied to, a `t_POLMOD` whose modulus is divisible by T (once mapped to a `FpX`), a suitable `t_RFRAC`. Returns z as an `FpXQ`, normalized.

`GEN Rg_to_Fq(GEN z, GEN T, GEN p)`, applies `Rg_to_Fp` if T is `NULL` and `Rg_to_FpXQ` otherwise.

`GEN RgX_to_FpXQX(GEN z, GEN T, GEN p)`, z a `t_POL`, returns the `FpXQ` obtained by applying `Rg_to_FpXQ` coefficientwise.

`GEN RgX_to_FqX(GEN z, GEN T, GEN p)`: let z be a `t_POL`; returns the `FqX` obtained by applying `Rg_to_Fq` coefficientwise.

`GEN Fq_to_FpXQ(GEN z, GEN T, GEN p /*unused*/)` if z is a `t_INT`, convert it to a constant polynomial in the variable of T , otherwise return z (shallow function).

`GEN Fq_red(GEN x, GEN T, GEN p)`, x a `ZX` or `t_INT`, reduce it to an `Fq` ($T = \text{NULL}$ is allowed iff x is a `t_INT`).

`GEN FqX_red(GEN x, GEN T, GEN p)`, x a `t_POL` whose coefficients are `ZXs` or `t_INTs`, reduce them to `Fqs`. (If $T = \text{NULL}$, as `FpXX_red(x, p)`.)

`GEN FqV_red(GEN x, GEN T, GEN p)`, x a vector of `ZXs` or `t_INTs`, reduce them to `Fqs`. (If $T = \text{NULL}$, only reduce components mod p to `FpXs` or `Fps`.)

`GEN FpXQ_red(GEN x, GEN T, GEN p)` x a `t_POL` whose coefficients are `t_INTs`, reduce them to `FpXQs`.

7.3.7 FpXQ.

GEN FpXQ_add(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_sub(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_mul(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_sqr(GEN x, GEN T, GEN p)

GEN FpXQ_div(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ_inv(GEN x, GEN T, GEN p) computes the inverse of x

GEN FpXQ_invsafe(GEN x, GEN T, GEN p), as FpXQ_inv, returning NULL if x is not invertible.

GEN FpXQ_pow(GEN x, GEN n, GEN T, GEN p) computes x^n .

GEN FpXQ_powu(GEN x, ulong n, GEN T, GEN p) computes x^n for small n .

In the following three functions the integer parameter `ord` can be given either as a positive `t_INT` N , or as its factorization matrix faN , or as a pair $[N, faN]$. The parameter may be omitted by setting it to NULL (the value is then $p^d - 1$, $d = \deg T$).

GEN FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) Let g be of order `ord` in the finite field $\mathbf{F}_p[X]/(T)$, return e such that $a^e = g$. If e does not exist, the result is undefined. Assumes that T is irreducible mod p .

GEN Fp_FpXQ_log(GEN a, GEN g, GEN ord, GEN T, GEN p) As FpXQ_log, a being a Fp.

GEN FpXQ_order(GEN a, GEN ord, GEN T, GEN p) returns the order of the FpXQ a . Assume that `ord` is a multiple of the order of a . Assume that T is irreducible mod p .

int FpXQ_issquare(GEN x, GEN T, GEN p) returns 1 if x is a square and 0 otherwise. Assumes that T is irreducible mod p .

GEN FpXQ_sqrt(GEN x, GEN T, GEN p) returns a square root of x . Return NULL if x is not a square.

GEN FpXQ_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN *zn) Let T be irreducible mod p and $q = p^{\deg T}$; returns NULL if a is not an n -th power residue mod p . Otherwise, returns an n -th root of a ; if `zn` is non-NULL set it to a primitive m -th root of 1 in \mathbf{F}_q , $m = \gcd(q - 1, n)$ allowing to compute all m solutions in \mathbf{F}_q of the equation $x^n = a$.

7.3.8 Fq.

GEN Fq_add(GEN x, GEN y, GEN T/*unused*/, GEN p)

GEN Fq_sub(GEN x, GEN y, GEN T/*unused*/, GEN p)

GEN Fq_mul(GEN x, GEN y, GEN T, GEN p)

GEN Fq_Fp_mul(GEN x, GEN y, GEN T, GEN p) multiplies the Fq x by the `t_INT` y .

GEN Fq_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the Fq x by the scalar y .

GEN Fq_halfve(GEN x, GEN T, GEN p) returns z such that $2z = x$ assuming such z exists.

GEN Fq_sqr(GEN x, GEN T, GEN p)

GEN Fq_neg(GEN x, GEN T, GEN p)

GEN Fq_neg_inv(GEN x, GEN T, GEN p) computes $-x^{-1}$
 GEN Fq_inv(GEN x, GEN pol, GEN p) computes x^{-1} , raising an error if x is not invertible.
 GEN Fq_invsafe(GEN x, GEN pol, GEN p) as Fq_inv, but returns NULL if x is not invertible.
 GEN Fq_div(GEN x, GEN y, GEN T, GEN p)
 GEN FqV_inv(GEN x, GEN T, GEN p) x being a vector of Fqs, return the vector of inverses of the $x[i]$. The routine uses Montgomery's trick, and involves a single inversion, plus $3(N - 1)$ multiplications for N entries. The routine is not stack-clean: $2N$ FpXQ are left on stack, besides the N in the result.
 GEN Fq_pow(GEN x, GEN n, GEN pol, GEN p) returns x^n .
 GEN Fq_powu(GEN x, ulong n, GEN pol, GEN p) returns x^n for small n .
 GEN Fq_log(GEN a, GEN g, GEN ord, GEN T, GEN p) as Fp_log or FpXQ_log.
 int Fq_issquare(GEN x, GEN T, GEN p) returns 1 if x is a square and 0 otherwise. Assumes that T is irreducible mod p and that p is prime; $T = \text{NULL}$ is forbidden unless x is an Fp.
 long Fq_ispower(GEN x, GEN n, GEN T, GEN p) returns 1 if x is a n -th power and 0 otherwise. Assumes that T is irreducible mod p and that p is prime; $T = \text{NULL}$ is forbidden unless x is an Fp.
 GEN Fq_sqrt(GEN x, GEN T, GEN p) returns a square root of x . Return NULL if x is not a square.
 GEN Fq_sqrtn(GEN a, GEN n, GEN T, GEN p, GEN *zn) as FpXQ_sqrtn.
 GEN FpXQ_charpoly(GEN x, GEN T, GEN p) returns the characteristic polynomial of x
 GEN FpXQ_minpoly(GEN x, GEN T, GEN p) returns the minimal polynomial of x
 GEN FpXQ_norm(GEN x, GEN T, GEN p) returns the norm of x
 GEN FpXQ_trace(GEN x, GEN T, GEN p) returns the trace of x
 GEN FpXQ_conjvec(GEN x, GEN T, GEN p) returns the vector of conjugates $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$ where n is the degree of T .
 GEN gener_FpXQ(GEN T, GEN p, GEN *po) returns a primitive root modulo (T, p) . T is an FpX assumed to be irreducible modulo the prime p . If po is not NULL it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.
 GEN gener_FpXQ_local(GEN T, GEN p, GEN L), L being a vector of primes dividing $p^{\deg T} - 1$, returns an element of $G := \mathbf{F}_p[x]/(T)$ which is a generator of the ℓ -Sylow of G for every ℓ in L . It is not necessary, and in fact slightly inefficient, to include $\ell = 2$, since 2 is treated separately in any case, i.e. the generator obtained is never a square if p is odd.
 GEN gener_Fq_local(GEN T, GEN p, GEN L) as pgener_Fp_local(p, L) if T is NULL, or gener_FpXQ_local (otherwise).
 GEN FpXQ_powers(GEN x, long n, GEN T, GEN p) returns $[x^0, \dots, x^n]$ as a t_VEC of FpXQs.
 GEN FpXQ_matrix_pow(GEN x, long m, long n, GEN T, GEN p), as FpXQ_powers($x, n - 1, T, p$), but returns the powers as a $m \times n$ matrix. Usually, we have $m = n = \deg T$.
 GEN FpXQ_autpow(GEN a, ulong n, GEN T, GEN p) computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_p[X]/T(X)$.

GEN FpXQ_autsum(GEN a, ulong n, GEN T, GEN p) a being a two-component vector, σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FpXQ_auttrace(GEN a, ulong n, GEN T, GEN p) a being a two-component vector, σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FpXQ_autpowers(GEN S, long n, GEN T, GEN p) returns $[x, S(x), S(S(x)), \dots, S^{(n)}(x)]$ as a t_VEC of FpXQs.

GEN FpXQM_autsum(GEN a, long n, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, returns the vector $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$ where $b = a[2]$ is a square matrix.

GEN FpX_FpXQ_eval(GEN f, GEN x, GEN T, GEN p) returns $f(x)$.

GEN FpX_FpXQV_eval(GEN f, GEN V, GEN T, GEN p) returns $f(x)$, assuming that V was computed by FpXQ_powers(x, n, T, p).

GEN FpXC_FpXQV_eval(GEN C, GEN V, GEN T, GEN p) applies FpX_FpXQV_eval to all elements of the vector C and returns a t_COL.

GEN FpXM_FpXQV_eval(GEN M, GEN V, GEN T, GEN p) applies FpX_FpXQV_eval to all elements of the matrix M .

7.3.9 FpXn. Let p a t_INT and T an FpX for p , both to be given in the function arguments; an FpXn object is an FpX whose degree is strictly less than n . They represent a class in $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$, in which all operations below take place. They can be seen as truncated power series.

GEN FpXn_mul(GEN x, GEN y, long n, GEN p) return $xy \pmod{X^n}$.

GEN FpXn_sqr(GEN x, long n, GEN p) return $x^2 \pmod{X^n}$.

GEN FpXn_inv(GEN x, long n, GEN p) return $1/x \pmod{X^n}$.

GEN FpXn_exp(GEN x, long n, GEN p) return $\exp(x)$ as a composition of formal power series. It is required that the valuation of x is positive and that $p > n$.

7.3.10 FpXC, FpXM.

GEN FpXC_center(GEN C, GEN p, GEN pov2)

GEN FpXM_center(GEN M, GEN p, GEN pov2)

7.3.11 FpXX, FpXY. Contrary to what the name implies, an FpXX is a t_POL whose coefficients are either t_INTs or FpXs. This reduces memory overhead at the expense of consistency. The prefix FpXY is an alias for FpXX when variables matters.

GEN FpXX_red(GEN z, GEN p), z a t_POL whose coefficients are either ZXs or t_INTs. Returns the t_POL equal to z with all components reduced modulo p.

GEN FpXX_renormalize(GEN x, long l), as normalizepol, where $l = \lg(x)$, in place.

GEN FpXX_add(GEN x, GEN y, GEN p) adds x and y.

GEN FpXX_sub(GEN x, GEN y, GEN p) returns $x - y$.

GEN FpXX_neg(GEN x, GEN p) returns $-x$.

GEN FpXX_Fp_mul(GEN x, GEN y, GEN p) multiplies the FpXX x by the Fp y.

GEN FpXX_FpX_mul(GEN x, GEN y, GEN p) multiplies the coefficients of the FpXX x by the FpX y.

GEN FpXX_mulu(GEN x, GEN y, GEN p) multiplies the FpXX x by the scalar y.

GEN FpXX_halve(GEN x, GEN p) returns z such that $2z = x$ assuming such z exists.

GEN FpXX_deriv(GEN P, GEN p) differentiates P with respect to the main variable.

GEN FpXX_integ(GEN P, GEN p) returns the primitive of P with respect to the main variable whose constant term is 0.

GEN FpXY_eval(GEN Q, GEN y, GEN x, GEN p) Q being an FpXY, i.e. a t_POL with Fp or FpX coefficients representing an element of $\mathbf{F}_p[X][Y]$. Returns the Fp $Q(x, y)$.

GEN FpXY_evalx(GEN Q, GEN x, GEN p) Q being an FpXY, returns the FpX $Q(x, Y)$, where Y is the main variable of Q.

GEN FpXY_evaly(GEN Q, GEN y, GEN p, long vx) Q an FpXY, returns the FpX $Q(X, y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_Fq_evaly(GEN Q, GEN y, GEN T, GEN p, long vx) Q an FpXY and y being an Fq, returns the FqX $Q(X, y)$, where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY_FpXQ_evalx(GEN Q, GEN x, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX $Q(x, Y)$, where Y is the first variable of Q.

GEN FpXY_FpXQV_evalx(GEN Q, GEN V, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX $Q(x, Y)$, where Y is the first variable of Q, assuming that V was computed by FpXQ_powers(x, n, T, p).

GEN FpXYQQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x being a FpXY, T being a FpX and S being a FpY, return $x^n \pmod{S, T, p}$.

7.3.12 FpXQX, FqX. Contrary to what the name implies, an FpXQX is a t_POL whose coefficients are Fqs. So the only difference between FqX and FpXQX routines is that $T = \text{NULL}$ is not allowed in the latter. (It was thought more useful to allow t_INT components than to enforce strict consistency, which would not imply any efficiency gain.)

7.3.12.1 Basic operations.

GEN FqX_add(GEN x, GEN y, GEN T, GEN p)

GEN FqX_Fq_add(GEN x, GEN y, GEN T, GEN p) adds the Fq y to the FqX x.

GEN FqX_Fq_sub(GEN x, GEN y, GEN T, GEN p) subtracts the Fq y to the FqX x.

GEN FqX_neg(GEN x, GEN T, GEN p)

GEN FqX_sub(GEN x, GEN y, GEN T, GEN p)

GEN FqX_mul(GEN x, GEN y, GEN T, GEN p)

GEN FqX_Fq_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the Fq y.

GEN FqX_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the FqX x by the scalar y.

GEN FqX_halve(GEN x, GEN T, GEN p) returns z such that $2z = x$ assuming such z exists.

GEN FqX_Fp_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the t_INT y.

GEN FqX_Fq_mul_to_monic(GEN x, GEN y, GEN T, GEN p) returns xy assuming the result is monic of the same degree as x (in particular $y \neq 0$).

GEN FpXQX_normalize(GEN z, GEN T, GEN p)

GEN FqX_normalize(GEN z, GEN T, GEN p) divides the FqX z by its leading term. The leading coefficient becomes 1 as a t_INT .

GEN FqX_sqr(GEN x, GEN T, GEN p)

GEN FqX_powu(GEN x, ulong n, GEN T, GEN p)

GEN FqX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *z)

GEN FqX_div(GEN x, GEN y, GEN T, GEN p)

GEN FqX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)

GEN FqX_rem(GEN x, GEN y, GEN T, GEN p)

GEN FqX_deriv(GEN x, GEN T, GEN p) returns the derivative of x. (This function is suitable for gerepilupto but not memory-clean.)

GEN FqX_integ(GEN x, GEN T, GEN p) returns the primitive of x. whose constant term is 0.

GEN FqX_translate(GEN P, GEN c, GEN T, GEN p) let c be an Fq defined modulo (p, T) , and let P be an FqX; returns the translated FqX of $P(X + c)$.

GEN FqX_gcd(GEN P, GEN Q, GEN T, GEN p) returns a (not necessarily monic) greatest common divisor of x and y .

GEN FqX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv) returns $d = \text{GCD}(x, y)$ (not necessarily monic), and sets $*u, *v$ to the Bezout coefficients such that $*ux + *vy = d$.

GEN FqX_halfgcd(GEN x, GEN y, GEN T, GEN p) returns a two-by-two FqXM M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN FqX_eval(GEN x, GEN y, GEN T, GEN p) evaluates the FqX x at the Fq y. The result is an Fq.

GEN FqXY_eval(GEN Q, GEN y, GEN x, GEN T, GEN p) Q an FqXY, i.e. a t_POL with Fq or FqX coefficients representing an element of $\mathbf{F}_q[X][Y]$. Returns the Fq $Q(x, y)$.

GEN FqXY_evalx(GEN Q, GEN x, GEN T, GEN p) Q being an FqXY, returns the FqX $Q(x, Y)$, where Y is the main variable of Q .

GEN random_FpXQX(long d, long v, GEN T, GEN p) returns a random FpXQX in variable v , of degree less than d .

GEN FpXQX_renormalize(GEN x, long lx)

GEN FpXQX_red(GEN z, GEN T, GEN p) z a t_POL whose coefficients are ZXs or t_INTs, reduce them to FpXQs.

GEN FpXQX_to_mod(GEN P, GEN T, GEN p) P being a FpXQX, converts each coefficient to a t_POLMOD with t_INTMOD coefficients.

GEN FqX_to_mod(GEN P, GEN T, GEN p) same but allow $T = \text{NULL}$.

GEN FpXQX_mul(GEN x, GEN y, GEN T, GEN p)

GEN Kronecker_to_FpXQX(GEN z, GEN T, GEN p). Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P , this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t) \pmod{(p, T(X))}$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y !

GEN FpXQX_FpXQ_mul(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_sqr(GEN x, GEN T, GEN p)

GEN FpXQX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *pr)

GEN FpXQX_div(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)

GEN FpXQX_rem(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_powu(GEN x, ulong n, GEN T, GEN p) returns x^n .

GEN FpXQX_digits(GEN x, GEN B, GEN T, GEN p)

GEN FpXQX_dotproduct(GEN x, GEN y, GEN T, GEN p) returns the scalar product of the coefficients of x and y .

GEN FpXQXV_FpXQX_fromdigits(GEN v, GEN B, GEN T, GEN p)

GEN FpXQX_invBarrett(GEN y, GEN T, GEN p) returns the Barrett inverse of the FpXQX y , namely a lift of $1/\text{polrecip}(y) + O(x^{\deg(y)-1})$.

GEN FpXQXV_prod(GEN V, GEN T, GEN p), V being a vector of FpXQX, returns their product.

GEN FpXQX_gcd(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)

GEN FpXQX_halfgcd(GEN x, GEN y, GEN T, GEN p)

GEN FpXQX_FpXQXQ_eval(GEN f, GEN x, GEN S, GEN T, GEN p) returns $f(x)$.

7.3.13 FpXQXn, FqXn.

A FpXQXn is a `t_FpXQX` which represents an element of the ring $(Fp[X]/T(X))[Y]/(Y^n)$, where T is a FpX.

GEN FpXQXn_sqr(GEN x, long n, GEN T, GEN p)

GEN FqXn_sqr(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_mul(GEN x, GEN y, long n, GEN T, GEN p)

GEN FqXn_mul(GEN x, GEN y, long n, GEN T, GEN p)

GEN FpXQXn_inv(GEN x, long n, GEN T, GEN p)

GEN FqXn_inv(GEN x, long n, GEN T, GEN p)

GEN FpXQXn_exp(GEN x, long n, GEN T, GEN p) return $\exp(x)$ as a composition of formal power series. It is required that the valuation of x is positive and that $p > n$.

GEN FqXn_exp(GEN x, long n, GEN T, GEN p)

7.3.14 FpXQXQ, FqXQ.

A FpXQXQ is a `t_FpXQX` which represents an element of the ring $(Fp[X]/T(X))[Y]/S(X, Y)$, where T is a FpX and S a FpXQX modulo T . A FqXQ is identical except that T is allowed to be NULL in which case S must be a FpX.

7.3.14.1 Preconditioned reduction.

For faster reduction, the modulus S can be replaced by an extended modulus, which is an FpXQXT, in all FpXQXQ- and FqXQ-classes functions, and in FpXQX_rem and FpXQX_divrem.

GEN FpXQX_get_red(GEN S, GEN T, GEN p) returns the extended modulus eS.

GEN FqX_get_red(GEN S, GEN T, GEN p) identical, but allow T to be NULL, in which case it returns FpX_get_red(S,p).

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_FpXQX_mod(GEN eS) returns the underlying modulus S.

GEN get_FpXQX_var(GEN eS) returns the variable number of the modulus.

GEN get_FpXQX_degree(GEN eS) returns the degree of the modulus.

Furthermore, ZXXT_to_FlxXT allows to convert an extended modulus for a FpXQX to an extended modulus for the corresponding FlxqX.

7.3.14.2 basic operations.

GEN FpXQX_FpXQXQV_eval(GEN f, GEN V, GEN S, GEN T, GEN p) returns $f(x)$, assuming that V was computed by FpXQXQ_powers(x, n, S, T, p).

GEN FpXQXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FpXQXs, returns $x * y^{-1}$ modulo S.

GEN FpXQXQ_inv(GEN x, GEN S, GEN T, GEN p), x and S being FpXQXs, returns x^{-1} modulo S.

GEN FpXQXQ_invsafe(GEN x, GEN S, GEN T, GEN p), as FpXQXQ_inv, returning NULL if x is not invertible.

GEN FpXQXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FpXQXs, returns xy modulo S.

GEN FpXQXQ_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FpXQXs, returns x^2 modulo S.

GEN FpXQXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FpXQXs, returns x^n modulo S.

GEN FpXQXQ_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FpXQXs, returns $[x^0, \dots, x^n]$ as a t_VEC of FpXQXs.

GEN FpXQXQ_halfFrobenius(GEN A, GEN S, GEN T, GEN p) returns $A(X)^{(q-1)/2} \pmod{S(X)}$ over the finite field \mathbf{F}_q defined by T and p, thus $q = p^n$ where n is the degree of T.

GEN FpXQXQ_minpoly(GEN x, GEN S, GEN T, GEN p), as FpXQ_minpoly

GEN FpXQXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FpXQXQ_powers(x, n - 1, S, T, p), but as an $m \times n$ matrix.

GEN FpXQXQ_autpow(GEN a, long n, GEN S, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X, Y), T(X)}$, returns $[\sigma^n(X), \sigma^n(Y)]$.

GEN FpXQXQ_autsum(GEN a, long n, GEN S, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$, $\sigma(Y) = a[2] \pmod{S(X, Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b\sigma(b) \dots \sigma^{n-1}(b)]$ where $b = a[3]$.

GEN FpXQXQ_auttrace(GEN a, long n, GEN S, GEN T, GEN p) σ being the automorphism defined by $\sigma(X) = X \pmod{T(X)}$, $\sigma(Y) = a[1] \pmod{S(X, Y), T(X)}$, returns the vector $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$ where $b = a[2]$.

GEN FqXQ_add(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns $x + y$ modulo S.

GEN FqXQ_sub(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns $x - y$ modulo S.

GEN FqXQ_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns xy modulo S.

GEN FqXQ_div(GEN x, GEN y, GEN S, GEN T, GEN p), x and S being FqXs, returns x/y modulo S.

GEN FqXQ_inv(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns x^{-1} modulo S.

GEN FqXQ_invsafe(GEN x, GEN S, GEN T, GEN p), as FqXQ_inv, returning NULL if x is not invertible.

GEN FqXQ_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns x^2 modulo S.

GEN FqXQ_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FqXs, returns x^n modulo S.

GEN FqXQ_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FqXs, returns $[x^0, \dots, x^n]$ as a t_VEC of FqXQs.

GEN FqXQ_matrix_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FqXQ_powers($x, n-1, S, T, p$), but as an $m \times n$ matrix.

GEN FqV_roots_to_pol(GEN V, GEN T, GEN p, long v), V being a vector of Fqs, returns the monic FqX $\prod_i(\text{pol}_x[v] - V[i])$.

7.3.14.3 Miscellaneous operations.

GEN init_Fq(GEN p, long n, long v) returns an irreducible polynomial of degree $n > 0$ over \mathbf{F}_p , in variable v.

int FqX_is_squarefree(GEN P, GEN T, GEN p)

GEN FpXQX_roots(GEN f, GEN T, GEN p) return the roots of f in $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FqX_roots(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX_factor(GEN f, GEN T, GEN p) same output convention as FpX_factor. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FqX_factor(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX_factor_squarefree(GEN f, GEN T, GEN p) squarefree factorization of f modulo (T, p) ; same output convention as FpX_factor_squarefree. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FqX_factor_squarefree(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX_ddf(GEN f, GEN T, GEN p) as FpX_ddf.

GEN FqX_ddf(GEN f, GEN T, GEN p) same but allow T = NULL.

long FpXQX_ddf_degree(GEN f, GEN XP, GEN T, GEN p), as FpX_ddf_degree.

GEN FpXQX_degfact(GEN f, GEN T, GEN p), as FpX_degfact.

GEN FqX_degfact(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX_split_part(GEN f, GEN T, GEN p) returns the largest totally split squarefree factor of f .

long FpXQX_isplayer(GEN f, ulong k, GEN T, GEN p, GEN *pt) return 1 if the FpXQX f is a k -th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

long FqX_isplayer(GEN f, ulong k, GEN T, GEN p, GEN *pt) same but allow T = NULL.

GEN FpX_factorff(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in $\mathbf{F}_p[X]$. Factor the FpX P over the finite field $\mathbf{F}_p[Y]/(T(Y))$. See FpX_factorff_irred if P is known to be irreducible of \mathbf{F}_p .

GEN FpX_rootsff(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in $\mathbf{F}_p[X]$. Returns the roots of the FpX P belonging to the finite field $\mathbf{F}_p[Y]/(T(Y))$.

GEN `FpX_factorff_irred`(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in $\mathbf{F}_p[X]$. Factors the *irreducible* FpX P over the finite field $\mathbf{F}_p[Y]/(T(Y))$ and returns the vector of irreducible FqXs factors (the exponents, being all equal to 1, are not included).

GEN `FpX_ffisom`(GEN P, GEN Q, GEN p). Assumes p prime, P, Q are ZXs, both irreducible mod p, and $\deg(P) \mid \deg(Q)$. Outputs a monomorphism between $\mathbf{F}_p[X]/(P)$ and $\mathbf{F}_p[X]/(Q)$, as a polynomial R such that $Q \mid P(R)$ in $\mathbf{F}_p[X]$. If P and Q have the same degree, it is of course an isomorphism.

void `FpX_ffintersect`(GEN P, GEN Q, long n, GEN p, GEN *SP, GEN *SQ, GEN MA, GEN MB) Assumes p is prime, P, Q are ZXs, both irreducible mod p, and n divides both the degree of P and Q. Compute SP and SQ such that the subfield of $\mathbf{F}_p[X]/(P)$ generated by SP and the subfield of $\mathbf{F}_p[X]/(Q)$ generated by SQ are isomorphic of degree n. The polynomials P and Q do not need to be of the same variable. If MA (resp. MB) is not NULL, it must be the matrix of the Frobenius map in $\mathbf{F}_p[X]/(P)$ (resp. $\mathbf{F}_p[X]/(Q)$).

GEN `FpXQ_ffisom_inv`(GEN S, GEN T, GEN p). Assumes p is prime, T a ZX, which is irreducible modulo p, S a ZX representing an automorphism of $\mathbf{F}_q := \mathbf{F}_p[X]/(T)$. (S(X) is the image of X by the automorphism.) Returns the inverse automorphism of S, in the same format, i.e. an FpX H such that $H(S) \equiv X$ modulo (T, p).

long `FpXQX_nbfact`(GEN S, GEN T, GEN p) returns the number of irreducible factors of the polynomial S over the finite field \mathbf{F}_q defined by T and p.

long `FpXQX_nbfact_Frobenius`(GEN S, GEN Xq, GEN T, GEN p) as `FpXQX_nbfact` where Xq is `FpXQX_Frobenius`(S, T, p).

long `FqX_nbfact`(GEN S, GEN T, GEN p) as above but accept T=NULL.

long `FpXQX_nbroots`(GEN S, GEN T, GEN p) returns the number of roots of the polynomial S over the finite field \mathbf{F}_q defined by T and p.

long `FqX_nbroots`(GEN S, GEN T, GEN p) as above but accept T=NULL.

GEN `FpXQX_Frobenius`(GEN S, GEN T, GEN p) returns $X^q \pmod{S(X)}$ over the finite field \mathbf{F}_q defined by T and p, thus $q = p^n$ where n is the degree of T.

7.3.15 Flx. Let p an understood ulong, assumed to be prime, to be given the function arguments; an Fl is an ulong belonging to $[0, p - 1]$, an Flx z is a `t_VECSMALL` representing a polynomial with small integer coefficients. Specifically `z[0]` is the usual codeword, `z[1] = evalvarn(v)` for some variable v, then the coefficients by increasing degree. An FlxX is a `t_POL` whose coefficients are Flxs.

In the following, an argument called sv is of the form `evalvarn(v)` for some variable number v.

7.3.15.1 Preconditioned reduction.

For faster reduction, the modulus T can be replaced by an extended modulus (FlxT) in all Flxq-classes functions, and in `Flx_divrem`.

GEN `Flx_get_red`(GEN T, ulong p) returns the extended modulus eT.

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN `get_Flx_mod`(GEN eT) returns the underlying modulus T.

GEN `get_Flx_var`(GEN eT) returns the variable number of the modulus.

GEN `get_Flx_degree`(GEN `eT`) returns the degree of the modulus.

Furthermore, `ZXT_to_FlxT` allows to convert an extended modulus for a `FpX` to an extended modulus for the corresponding `Flx`.

7.3.15.2 Basic operations.

`ulong Flx_lead`(GEN `x`) returns the leading coefficient of x as a `ulong` (return 0 for the zero polynomial).

`ulong Flx_constant`(GEN `x`) returns the constant coefficient of x as a `ulong` (return 0 for the zero polynomial).

GEN `Flx_red`(GEN `z`, `ulong p`) converts from `zx` with non-negative coefficients to `Flx` (by reducing them mod p).

`int Flx_equal1`(GEN `x`) returns 1 (true) if the `Flx` x is equal to 1, 0 (false) otherwise.

`int Flx_equal`(GEN `x`, GEN `y`) returns 1 (true) if the `Flx` x and y are equal, and 0 (false) otherwise.

GEN `Flx_copy`(GEN `x`) returns a copy of `x`.

GEN `Flx_add`(GEN `x`, GEN `y`, `ulong p`)

GEN `Flx_Fl_add`(GEN `y`, `ulong x`, `ulong p`)

GEN `Flx_neg`(GEN `x`, `ulong p`)

GEN `Flx_neg_inplace`(GEN `x`, `ulong p`), same as `Flx_neg`, in place (`x` is destroyed).

GEN `Flx_sub`(GEN `x`, GEN `y`, `ulong p`)

GEN `Flx_half`(GEN `x`, `ulong p`) returns z such that $2z = x$ modulo p assuming such z exists.

GEN `Flx_mul`(GEN `x`, GEN `y`, `ulong p`)

GEN `Flxn_mul`(GEN `a`, GEN `b`, `long n`, `ulong p`) returns ab modulo X^n .

GEN `Flxn_inv`(GEN `a`, `long n`, `ulong p`) returns $1/a$ modulo X^n .

GEN `Flx_Fl_mul`(GEN `y`, `ulong x`, `ulong p`)

GEN `Flx_double`(GEN `y`, `ulong p`) returns $2y$.

GEN `Flx_triple`(GEN `y`, `ulong p`) returns $3y$.

GEN `Flx_mulu`(GEN `y`, `ulong x`, `ulong p`) as `Flx_Fl_mul` but do not assume that $x < p$.

GEN `Flx_Fl_mul_to_monic`(GEN `y`, `ulong x`, `ulong p`) returns yx assuming the result is monic of the same degree as y (in particular $x \neq 0$).

GEN `Flx_sqr`(GEN `x`, `ulong p`)

GEN `Flx_powu`(GEN `x`, `ulong n`, `ulong p`) return x^n .

GEN `Flx_divrem`(GEN `x`, GEN `y`, `ulong p`, GEN `*pr`)

GEN `Flx_div`(GEN `x`, GEN `y`, `ulong p`)

GEN `Flx_rem`(GEN `x`, GEN `y`, `ulong p`)

GEN `Flx_deriv`(GEN `z`, `ulong p`)

GEN `Flx_translate1`(GEN `P`, `ulong p`) return $P(x + 1)$

GEN Flx_diff1(GEN P, ulong p) return $P(x+1) - P(x)$

GEN Flx_digits(GEN x, GEN B, ulong p) returns a vector of Flx $[c_0, \dots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^n c_i B^i$.

GEN FlxV_Flx_fromdigits(GEN v, GEN B, ulong p) where $v = [c_0, \dots, c_n]$ is a vector of Flx, returns $\sum_{i=0}^n c_i B^i$.

GEN Flx_Frobenius(GEN T, ulong p)

GEN Flx_matFrobenius(GEN T, ulong p)

GEN Flx_gcd(GEN a, GEN b, ulong p) returns a (not necessarily monic) greatest common divisor of x and y .

GEN Flx_halfgcd(GEN x, GEN y, ulong p) returns a two-by-two FlxM M with determinant ± 1 such that the image (a, b) of (x, y) by M has the property that $\deg a \geq \frac{\deg x}{2} > \deg b$.

GEN Flx_extgcd(GEN a, GEN b, ulong p, GEN *ptu, GEN *ptv)

GEN Flx_roots(GEN f, ulong p) returns the vector of roots of f (without multiplicity, as a t_VEC SMALL). Assumes that p is prime.

ulong Flx_oneroot(GEN f, ulong p) returns one root $0 \leq r < p$ of the Flx f in $\mathbf{Z}/p\mathbf{Z}$. Return p if no root exists. Assumes that p is prime.

ulong Flx_oneroot_split(GEN f, ulong p) as Flx_oneroot but assume f is totally split.

long Flx_ispower(GEN f, ulong k, ulong p, GEN *pt) return 1 if the Flx f is a k -th power, 0 otherwise. If pt is not NULL, set it to g such that $g^k = f$.

GEN Flx_factor(GEN f, ulong p)

GEN Flx_ddf(GEN f, ulong p)

GEN Flx_factor_squarefree(GEN f, ulong p) returns the squarefree factorization of f modulo p . This is a vector $[u_1, \dots, u_k]$ of pairwise coprime Flx such that $u_k \neq 1$ and $f = \prod u_i^i$. Shallow function.

GEN Flx_mod_Xn1(GEN T, ulong n, ulong p) return T modulo $(X^n + 1, p)$. Shallow function.

GEN Flx_mod_Xnm1(GEN T, ulong n, ulong p) return T modulo $(X^n - 1, p)$. Shallow function.

GEN Flx_degfact(GEN f, ulong p) as FpX_degfact.

GEN Flx_factorff_irred(GEN P, GEN Q, ulong p) as FpX_factorff_irred.

GEN Flx_rootsff(GEN P, GEN T, ulong p) as FpX_rootsff.

GEN Flx_ffisom(GEN P, GEN Q, ulong l) as FpX_ffisom.

7.3.15.3 Miscellaneous operations.

GEN `pol0_Flx(long sv)` returns a zero Flx in variable v .

GEN `zero_Flx(long sv)` alias for `pol0_Flx`

GEN `pol1_Flx(long sv)` returns the unit Flx in variable v .

GEN `polx_Flx(long sv)` returns the variable v as degree 1 Flx.

GEN `monomial_Flx(ulong a, long d, long sv)` returns the Flx aX^d in variable v .

GEN `Flx_normalize(GEN z, ulong p)`, as `FpX_normalize`.

GEN `Flx_rescale(GEN P, ulong h, ulong p)` returns $h^{\deg(P)}P(x/h)$, P is a Flx and h is a non-zero integer.

GEN `random_Flx(long d, long sv, ulong p)` returns a random Flx in variable v , of degree less than d .

GEN `Flx_recip(GEN x)`, returns the reciprocal polynomial

`ulong Flx_resultant(GEN a, GEN b, ulong p)`, returns the resultant of a and b

`ulong Flx_extresultant(GEN a, GEN b, ulong p, GEN *ptU, GEN *ptV)` given two Flx a and b , returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set).

GEN `Flx_invBarrett(GEN T, ulong p)`, returns the Barrett inverse M of T defined by $M(x) \times x^n T(1/x) \equiv 1 \pmod{x^{n-1}}$ where n is the degree of T .

GEN `Flx_renormalize(GEN x, long l)`, as `FpX_renormalize`, where $l = \lg(x)$, in place.

GEN `Flx_shift(GEN T, long n)` returns $T * x^n$ if $n \geq 0$, and $T \setminus x^{-n}$ otherwise.

`long Flx_val(GEN x)` returns the valuation of x , i.e. the multiplicity of the 0 root.

`long Flx_valrem(GEN x, GEN *Z)` as `RgX_valrem`, returns the valuation of x . In particular, if the valuation is 0, set $*Z$ to x , not a copy.

GEN `Flx_div_by_X_x(GEN A, ulong a, ulong p, ulong *rem)`, returns the Euclidean quotient of the Flx A by $X - a$, and sets `rem` to the remainder $A(a)$.

`ulong Flx_eval(GEN x, ulong y, ulong p)`, as `FpX_eval`.

`ulong Flx_eval_pre(GEN x, ulong y, ulong p, ulong pi)`, as `Flx_eval`, assuming pi is the pseudo inverse of p .

`ulong Flx_eval_powers_pre(GEN P, GEN y, ulong p, ulong pi)`. Let y be the `t_VECSMALL` $(1, a, \dots, a^n)$, where n is the degree of the Flx P , return $P(a)$, assuming pi is the pseudo inverse of p .

GEN `Flx_Flv_multieval(GEN P, GEN v, ulong p)` returns the vector $[P(v[1]), \dots, P(v[n])]$ as a Flv.

`ulong Flx_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of the coefficients of x and y .

GEN `Flx_deflate(GEN P, long d)` assuming P is a polynomial of the form $Q(X^d)$, return Q .

GEN `Flx_splitting(GEN P, long k)`, as `RgX_splitting`.

GEN `Flx_inflate(GEN P, long d)` returns $P(X^d)$.

```

int Flx_is_squarefree(GEN z, ulong p)

int Flx_is_irred(GEN f, ulong p), as FpX_is_irred.

int Flx_is_smooth(GEN f, long r, ulong p) return 1 if all irreducible factors of  $f$  are of degree
at most  $r$ , 0 otherwise.

long Flx_nbroots(GEN f, ulong p), as FpX_nbroots.

long Flx_nbfact(GEN z, ulong p), as FpX_nbfact.

long Flx_nbfact_Frobenius(GEN f, GEN XP, ulong p), as FpX_nbfact_Frobenius.

GEN Flx_degfact(GEN f, ulong p), as FpX_degfact.

GEN Flx_nbfact_by_degree(GEN z, long *nb, ulong p) Assume that the Flx  $z$  is squarefree
mod the prime  $p$ . Returns a t_VECSMALL  $D$  with  $\deg z$  entries, such that  $D[i]$  is the number of
irreducible factors of degree  $i$ . Set  $nb$  to the total number of irreducible factors (the sum of the
 $D[i]$ ).

void Flx_ffintersect(GEN P, GEN Q, long n, ulong p, GEN*SP, GEN*SQ, GEN MA, GEN MB)
,
as FpX_ffintersect

GEN Flv_polint(GEN x, GEN y, ulong p, long sv) as FpV_polint, returning an Flx in variable
 $v$ .

GEN Flv_Flm_polint(GEN x, GEN V, ulong p, long sv) equivalent (but faster) to applying
Flv_polint( $x, \dots$ ) to all the elements of the vector  $V$  (thus, returns a FlxV).

GEN Flv_invVandermonde(GEN L, ulong d, ulong p)  $L$  being a Flv of length  $n$ , return the
inverse  $M$  of the Vandermonde matrix attached to the elements of  $L$ , multiplied by  $d$ . If  $A$  is a Flv
and  $B = MA$ , then the polynomial  $P = \sum_{i=1}^n B[i]X^{i-1}$  verifies  $P(L[i]) = dA[i]$  for  $1 \leq i \leq n$ .

GEN Flv_roots_to_pol(GEN a, ulong p, long sv) as FpV_roots_to_pol returning an Flx in
variable  $v$ .

7.3.16 FlxV. See FpXV operations.

GEN FlxV_Flc_mul(GEN V, GEN W, ulong p), as FpXV_FpC_mul.

GEN FlxV_red(GEN V, ulong p) reduces each components with Flx_red.

GEN FlxV_prod(GEN V, ulong p),  $V$  being a vector of Flx, returns their product.

ulong FlxC_eval_powers_pre(GEN x, GEN y, ulong p, ulong pi) apply Flx_eval_powers_pre
to all elements of  $x$ .

GEN FlxC_neg(GEN x, ulong p)

GEN FlxC_sub(GEN x, GEN y, ulong p)

GEN zero_FlxC(long n, long sv)

```

7.3.17 FlxM. See FpXM operations.

`ulong FlxM_eval_powers_pre(GEN M, GEN y, ulong p, ulong pi)` this function applies `FlxC_eval_powers_pre` to all entries of `M`.

`GEN FlxM_neg(GEN x, ulong p)`

`GEN FlxM_sub(GEN x, GEN y, ulong p)`

`GEN zero_FlxM(long r, long c, long sv)`

7.3.18 FlxT. See FpXT operations.

`GEN FlxT_red(GEN V, ulong p)` reduces each leaf with `Flx_red`.

7.3.19 Flxq. See FpXQ operations.

`GEN Flxq_add(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sub(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_mul(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_sqr(GEN y, GEN T, ulong p)`

`GEN Flxq_inv(GEN x, GEN T, ulong p)`

`GEN Flxq_invsafe(GEN x, GEN T, ulong p)`

`GEN Flxq_div(GEN x, GEN y, GEN T, ulong p)`

`GEN Flxq_pow(GEN x, GEN n, GEN T, ulong p)`

`GEN Flxq_powu(GEN x, ulong n, GEN T, ulong p)`

`GEN Flxq_pow_init(GEN x, GEN n, long k, GEN T, ulong p)`

`GEN Flxq_pow_table(GEN R, GEN n, GEN T, ulong p)`

`GEN Flxq_powers(GEN x, long n, GEN T, ulong p)`

`GEN Flxq_matrix_pow(GEN x, long m, long n, GEN T, ulong p)`, see `FpXQ_matrix_pow`.

`GEN Flxq_autpow(GEN a, long n, GEN T, ulong p)` see `FpXQ_autpow`.

`GEN Flxq_autsum(GEN a, long n, GEN T, ulong p)` see `FpXQ_autsum`.

`GEN Flxq_auttrace(GEN a, ulong n, GEN T, ulong p)` see `FpXQ_auttrace`.

`GEN Flxq_ffisom_inv(GEN S, GEN T, ulong p)`, as `FpXQ_ffisom_inv`.

`GEN Flx_Flxq_eval(GEN f, GEN x, GEN T, ulong p)` returns $f(x)$.

`GEN Flx_FlxqV_eval(GEN f, GEN x, GEN T, ulong p)`, see `FpX_FpXqV_eval`.

`GEN FlxqV_roots_to_pol(GEN V, GEN T, ulong p, long v)` as `FqV_roots_to_pol` returning an `FlxqX` in variable v .

`int Flxq_issquare(GEN x, GEN T, ulong p)` returns 1 if x is a square and 0 otherwise. Assume that T is irreducible mod p .

`int Flxq_is2npower(GEN x, long n, GEN T, ulong p)` returns 1 if x is a 2^n -th power and 0 otherwise. Assume that T is irreducible mod p .

GEN Flxq_order(GEN a, GEN ord, GEN T, ulong p) as FpXQ_order.
 GEN Flxq_log(GEN a, GEN g, GEN ord, GEN T, ulong p) as FpXQ_log
 GEN Flxq_sqrtn(GEN x, GEN n, GEN T, ulong p, GEN *zn) as FpXQ_sqrtn.
 GEN Flxq_sqrt(GEN x, GEN T, ulong p) returns a square root of x. Return NULL if x is not a square.
 GEN Flxq_lroot(GEN a, GEN T, ulong p) returns x such that $x^p = a$.
 GEN Flxq_lroot_fast(GEN a, GEN V, GEN T, ulong p) assuming that $V = \text{Flxq_powers}(s, p-1, T, p)$ where $s(x)^p \equiv x \pmod{T(x), p}$, returns b such that $b^p = a$. Only useful if p is less than the degree of T .
 GEN Flxq_charpoly(GEN x, GEN T, ulong p) returns the characteristic polynomial of x
 GEN Flxq_minpoly(GEN x, GEN T, ulong p) returns the minimal polynomial of x
 ulong Flxq_norm(GEN x, GEN T, ulong p) returns the norm of x
 ulong Flxq_trace(GEN x, GEN T, ulong p) returns the trace of x
 GEN Flxq_conjvec(GEN x, GEN T, ulong p) returns the conjugates $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$ where n is the degree of T .
 GEN gener_Flxq(GEN T, ulong p, GEN *po) returns a primitive root modulo (T, p) . T is an Flx assumed to be irreducible modulo the prime p . If po is not NULL it is set to $[o, fa]$, where o is the order of the multiplicative group of the finite field, and fa is its factorization.

7.3.20 FlxX. See FpXX operations.

GEN pol1_FlxX(long vX, long sx) returns the unit FlxX as a t_POL in variable vX which only coefficient is pol1_Flx(sx).
 GEN polx_FlxX(long vX, long sx) returns the variable X as a degree 1 t_POL with Flx coefficients in the variable x .
 long FlxY_degreex(GEN P) return the degree of P with respect to the secondary variable.
 GEN FlxX_add(GEN P, GEN Q, ulong p)
 GEN FlxX_sub(GEN P, GEN Q, ulong p)
 GEN FlxX_Fl_mul(GEN x, ulong y, ulong p)
 GEN FlxX_double(GEN x, ulong p)
 GEN FlxX_triple(GEN x, ulong p)
 GEN FlxX_neg(GEN x, ulong p)
 GEN FlxX_Flx_add(GEN x, GEN y, ulong p)
 GEN FlxX_Flx_sub(GEN x, GEN y, ulong p)
 GEN FlxX_Flx_mul(GEN x, GEN y, ulong p)
 GEN FlxY_Flx_div(GEN x, GEN y, ulong p) divides the coefficients of x by y using Flx_div.
 GEN FlxX_deriv(GEN P, ulong p) returns the derivative of P with respect to the main variable.

GEN FlxY_evalx(GEN P, ulong z, ulong p) P being an FlxY, returns the Flx $P(z, Y)$, where Y is the main variable of P .

GEN FlxY_Flx_translate(GEN P, GEN f, ulong p) P being an FlxY and f being an Flx, return $(P(x, Y + f(x)))$, where Y is the main variable of P .

ulong FlxY_evalx_powers_pre(GEN P, GEN xp, ulong p, ulong pi), xp being the vector $[1, x, \dots, x^n]$, where n is larger or equal to the degree of P in X , return $P(x, Y)$, where Y is the main variable of Q , assuming pi is the pseudo inverse of p .

ulong FlxY_eval_powers_pre(GEN P, GEN xp, GEN yp, ulong p, ulong pi), xp being the vector $[1, x, \dots, x^n]$, where n is larger or equal to the degree of P in X and yp being the vector $[1, y, \dots, y^m]$, where m is larger or equal to the degree of P in Y return $P(x, y)$, assuming pi is the pseudo inverse of p .

GEN FlxY_Flxq_evalx(GEN x, GEN y, GEN T, ulong p) as FpXY_FpXQ_evalx.

GEN FlxY_FlxqV_evalx(GEN x, GEN V, GEN T, ulong p) as FpXY_FpXQV_evalx.

GEN FlxX_renormalize(GEN x, long l), as normalizepol, where $l = \lg(x)$, in place.

GEN FlxX_resultant(GEN u, GEN v, ulong p, long sv) Returns $\text{Res}_X(u, v)$, which is an Flx. The coefficients of u and v are assumed to be in the variable v .

GEN Flx_FlxY_resultant(GEN a, GEN b, ulong p) Returns $\text{Res}_x(a, b)$, which is an Flx in the main variable of b .

GEN FlxX_shift(GEN a, long n, long sv), as RgX_shift_shallow, where v is the secondary variable.

GEN FlxX_swap(GEN x, long n, long ws), as RgXY_swap.

GEN FlxYqq_pow(GEN x, GEN n, GEN S, GEN T, ulong p), as FpXYQQ_pow.

7.3.21 FlxqX. See FpXQX operations.

7.3.21.1 Preconditioned reduction.

For faster reduction, the modulus S can be replaced by an extended modulus, which is an FlxqXT, in all FlxqXQ-classes functions, and in FlxqX_rem and FlxqX_divrem.

GEN FlxqX_get_red(GEN S, GEN T, ulong p) returns the extended modulus eS .

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_FlxqX_mod(GEN eS) returns the underlying modulus S .

GEN get_FlxqX_var(GEN eS) returns the variable number of the modulus.

GEN get_FlxqX_degree(GEN eS) returns the degree of the modulus.

7.3.21.2 basic functions.

GEN `random_FlxqX(long d, long v, GEN T, ulong p)` returns a random `FlxqX` in variable `v`, of degree less than `d`.

GEN `zxX_to_Kronecker(GEN P, GEN Q)` assuming $P(X, Y)$ is a polynomial of degree in X strictly less than n , returns $P(X, X^{2^{*n-1}})$, the Kronecker form of P .

GEN `Kronecker_to_FlxqX(GEN z, GEN T, ulong p)`. Let $n = \deg T$ and let $P(X, Y) \in \mathbf{Z}[X, Y]$ lift a polynomial in $K[Y]$, where $K := \mathbf{F}_p[X]/(T)$ and $\deg_X P < 2n-1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2^{*n-1}})$ be a Kronecker form of P , this function returns $Q \in \mathbf{Z}[X, t]$ such that Q is congruent to $P(X, t) \pmod{(p, T(X))}$, $\deg_X Q < n$, and all coefficients are in $[0, p[$. Not stack-clean. Note that t need not be the same variable as Y !

GEN `FlxqX_red(GEN z, GEN T, ulong p)`

GEN `FlxqX_normalize(GEN z, GEN T, ulong p)`

GEN `FlxqX_mul(GEN x, GEN y, GEN T, ulong p)`

GEN `FlxqX_Flxq_mul(GEN P, GEN U, GEN T, ulong p)`

GEN `FlxqX_Flxq_mul_to_monic(GEN P, GEN U, GEN T, ulong p)` returns $P * U$ assuming the result is monic of the same degree as P (in particular $U \neq 0$).

GEN `FlxqX_sqr(GEN x, GEN T, ulong p)`

GEN `FlxqX_powu(GEN x, ulong n, GEN T, ulong p)`

GEN `FlxqX_divrem(GEN x, GEN y, GEN T, ulong p, GEN *pr)`

GEN `FlxqX_div(GEN x, GEN y, GEN T, ulong p)`

GEN `FlxqX_rem(GEN x, GEN y, GEN T, ulong p)`

GEN `FlxqX_invBarrett(GEN T, GEN Q, ulong p)`

GEN `FlxqX_gcd(GEN x, GEN y, ulong p)` returns a (not necessarily monic) greatest common divisor of x and y .

GEN `FlxqX_extgcd(GEN x, GEN y, GEN T, ulong p, GEN *ptu, GEN *ptv)`

GEN `FlxqX_halfgcd(GEN x, GEN y, GEN T, ulong p)`, see `FpX_halfgcd`.

GEN `FlxqXV_prod(GEN V, GEN T, ulong p)`

GEN `FlxqX_safegcd(GEN P, GEN Q, GEN T, ulong p)` Returns the *monic* GCD of P and Q if Euclid's algorithm succeeds and `NULL` otherwise. In particular, if p is not prime or T is not irreducible over $\mathbf{F}_p[X]$, the routine may still be used (but will fail if non-invertible leading terms occur).

GEN `FlxqX_dotproduct(GEN x, GEN y, GEN T, ulong p)` returns the scalar product of the coefficients of x and y .

`long FlxqX_is_squarefree(GEN S, GEN T, ulong p)`, as `FpX_is_squarefree`.

`long FlxqX_ispower(GEN f, ulong k, GEN T, ulong p, GEN *pt)` return 1 if the `FlxqX` f is a k -th power, 0 otherwise. If `pt` is not `NULL`, set it to g such that $g^k = f$.

GEN `FlxqX_Frobenius(GEN S, GEN T, ulong p)`, as `FpXQX_Frobenius`

GEN FlxqX_roots(GEN f, GEN T, ulong p) return the roots of f in $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FlxqX_factor(GEN f, GEN T, ulong p) return the factorization of f over $\mathbf{F}_p[X]/(T)$. Assumes p is prime and T irreducible in $\mathbf{F}_p[X]$.

GEN FlxqX_factor_squarefree(GEN f, GEN T, ulong p) returns the squarefree factorization of f , see FpX_factor_squarefree.

GEN FlxqX_ddf(GEN f, GEN T, ulong p) as FpX_ddf.

long FlxqX_ddf_degree(GEN f, GEN XP, GEN T, GEN p), as FpX_ddf_degree.

GEN FlxqX_degfact(GEN f, GEN T, ulong p), as FpX_degfact.

long FlxqX_nbroots(GEN S, GEN T, ulong p), as FpX_nbroots.

long FlxqX_nbfact(GEN S, GEN T, ulong p), as FpX_nbfact.

long FlxqX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, ulong p), as FpX_nbfact_Frobenius.

GEN FlxqX_FlxqXQ_eval(GEN Q, GEN x, GEN S, GEN T, ulong p) as FpX_FpXQ_eval.

GEN FlxqX_FlxqXQV_eval(GEN P, GEN V, GEN S, GEN T, ulong p) as FpX_FpXQV_eval.

7.3.22 FlxqXQ. See FpXQXQ operations.

GEN FlxqXQ_mul(GEN x, GEN y, GEN S, GEN T, ulong p)

GEN FlxqXQ_sqr(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ_inv(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ_div(GEN x, GEN y, GEN S, GEN T, ulong p)

GEN FlxqXQ_pow(GEN x, GEN n, GEN S, GEN T, ulong p)

GEN FlxqXQ_powu(GEN x, ulong n, GEN S, GEN T, ulong p)

GEN FlxqXQ_powers(GEN x, long n, GEN S, GEN T, ulong p)

GEN FlxqXQ_matrix_pow(GEN x, long n, long m, GEN S, GEN T, ulong p)

GEN FlxqXQ_autpow(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_autpow

GEN FlxqXQ_autsum(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_autsum

GEN FlxqXQ_auttrace(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ_auttrace

GEN FlxqXQ_halfFrobenius(GEN A, GEN S, GEN T, ulong p), as FpXQXQ_halfFrobenius

GEN FlxqXQ_minpoly(GEN x, GEN S, GEN T, ulong p), as FpXQ_minpoly

7.3.23 F2x. An F2x z is a `t_VECSMALL` representing a polynomial over $\mathbf{F}_2[X]$. Specifically $z[0]$ is the usual codeword, $z[1] = \text{evalvarn}(v)$ for some variable v and the coefficients are given by the bits of remaining words by increasing degree.

7.3.23.1 Preconditioned reduction.

For faster reduction, the modulus T can be replaced by an extended modulus ($\text{Flx}T$) in all Flxq -classes functions, and in Flx_divrem .

`GEN F2x_get_red(GEN T)` returns the extended modulus eT .

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_F2x_mod(GEN eT)` returns the underlying modulus T .

`GEN get_F2x_var(GEN eT)` returns the variable number of the modulus.

`GEN get_F2x_degree(GEN eT)` returns the degree of the modulus.

7.3.23.2 Basic operations.

`ulong F2x_coeff(GEN x, long i)` returns the coefficient $i \geq 0$ of x .

`void F2x_clear(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 0.

`void F2x_flip(GEN x, long i)` adds 1 to the coefficient $i \geq 0$ of x .

`void F2x_set(GEN x, long i)` sets the coefficient $i \geq 0$ of x to 1.

`GEN F2x_copy(GEN x)`

`GEN Flx_to_F2x(GEN x)`

`GEN Z_to_F2x(GEN x, long v)`

`GEN ZX_to_F2x(GEN x)`

`GEN F2v_to_F2x(GEN x, long sv)`

`GEN F2x_to_Flx(GEN x)`

`GEN F2x_to_F2xX(GEN x, long sv)`

`GEN F2x_to_ZX(GEN x)`

`GEN pol0_F2x(long sv)` returns a zero F2x in variable v .

`GEN zero_F2x(long sv)` alias for `pol0_F2x`.

`GEN pol1_F2x(long sv)` returns the F2x in variable v constant to 1.

`GEN polx_F2x(long sv)` returns the variable v as degree 1 F2x.

`GEN monomial_F2x(long d, long sv)` returns the F2x X^d in variable v .

`GEN random_F2x(long d, long sv)` returns a random F2x in variable v , of degree less than d .

`long F2x_degree(GEN x)` returns the degree of the F2x x . The degree of 0 is defined as -1 .

`int F2x_equal1(GEN x)`

`int F2x_equal(GEN x, GEN y)`

GEN F2x_1_add(GEN y) returns $y+1$ where y is a Flx.
 GEN F2x_add(GEN x, GEN y)
 GEN F2x_mul(GEN x, GEN y)
 GEN F2x_sqr(GEN x)
 GEN F2x_divrem(GEN x, GEN y, GEN *pr)
 GEN F2x_rem(GEN x, GEN y)
 GEN F2x_div(GEN x, GEN y)
 GEN F2x_renormalize(GEN x, long lx)
 GEN F2x_deriv(GEN x)
 GEN F2x_deflate(GEN x, long d)
 ulong F2x_eval(GEN P, ulong u) returns $P(u)$.
 void F2x_shift(GEN x, long d) as RgX_shift
 void F2x_even_odd(GEN P, GEN *pe, GEN *po) as RgX_even_odd
 long F2x_valrem(GEN x, GEN *Z)
 GEN F2x_extgcd(GEN a, GEN b, GEN *ptu, GEN *ptv)
 GEN F2x_gcd(GEN a, GEN b)
 GEN F2x_halfgcd(GEN a, GEN b)
 int F2x_issquare(GEN x) returns 1 if x is a square of a F2x and 0 otherwise.
 int F2x_is_irred(GEN f), as FpX_is_irred.
 GEN F2x_degfact(GEN f) as FpX_degfact.
 GEN F2x_sqrt(GEN x) returns the squareroot of x , assuming x is a square of a F2x.
 GEN F2x_Frobenius(GEN T)
 GEN F2x_matFrobenius(GEN T)
 GEN F2x_factor(GEN f)
 GEN F2x_factor_squarefree(GEN f)
 GEN F2x_ddf(GEN f)

7.3.24 F2xq. See FpXQ operations.

GEN F2xq_mul(GEN x, GEN y, GEN T)

GEN F2xq_sqr(GEN x, GEN T)

GEN F2xq_div(GEN x, GEN y, GEN T)

GEN F2xq_inv(GEN x, GEN T)

GEN F2xq_invsafe(GEN x, GEN T)

GEN F2xq_pow(GEN x, GEN n, GEN T)

GEN F2xq_powu(GEN x, ulong n, GEN T)

GEN F2xq_pow_init(GEN x, GEN n, long k, GEN T)

GEN F2xq_pow_table(GEN R, GEN n, GEN T)

ulong F2xq_trace(GEN x, GEN T)

GEN F2xq_conjvec(GEN x, GEN T) returns the vector of conjugates $[x, x^2, x^{2^2}, \dots, x^{2^{n-1}}]$ where n is the degree of T .

GEN F2xq_log(GEN a, GEN g, GEN ord, GEN T)

GEN F2xq_order(GEN a, GEN ord, GEN T)

GEN F2xq_Artin_Schreier(GEN a, GEN T) returns a solution of $x^2 + x = a$, assuming it exists.

GEN F2xq_sqrt(GEN a, GEN T)

GEN F2xq_sqrt_fast(GEN a, GEN s, GEN T) assuming that $s^2 \equiv x \pmod{T(x)}$, computes $b \equiv a(s) \pmod{T}$ so that $b^2 = a$.

GEN F2xq_sqrtn(GEN a, GEN n, GEN T, GEN *zeta)

GEN gener_F2xq(GEN T, GEN *po)

GEN F2xq_powers(GEN x, long n, GEN T)

GEN F2xq_matrix_pow(GEN x, long m, long n, GEN T)

GEN F2x_F2xq_eval(GEN f, GEN x, GEN T)

GEN F2x_F2xqV_eval(GEN f, GEN x, GEN T), see FpX_FpXQV_eval.

GEN F2xq_autpow(GEN a, long n, GEN T) computes $\sigma^n(X)$ assuming $a = \sigma(X)$ where σ is an automorphism of the algebra $\mathbf{F}_2[X]/T(X)$.

7.3.25 F2xqV, F2xqM.. See FqV, FqM operations.

GEN F2xqM_F2xqC_gauss(GEN a, GEN b, GEN T)
GEN F2xqM_F2xqC_invimage(GEN a, GEN b, GEN T)
GEN F2xqM_F2xqC_mul(GEN a, GEN b, GEN T)
GEN F2xqM_deplin(GEN x, GEN T)
GEN F2xqM_det(GEN a, GEN T)
GEN F2xqM_gauss(GEN a, GEN b, GEN T)
GEN F2xqM_image(GEN x, GEN T)
GEN F2xqM_indexrank(GEN x, GEN T)
GEN F2xqM_inv(GEN a, GEN T)
GEN F2xqM_invimage(GEN a, GEN b, GEN T)
GEN F2xqM_ker(GEN x, GEN T)
GEN F2xqM_mul(GEN a, GEN b, GEN T)
long F2xqM_rank(GEN x, GEN T)
GEN F2xqM_suppl(GEN x, GEN T)
GEN matid_F2xqM(long n, GEN T)

7.3.26 F2xX.. See FpXX operations.

GEN ZXX_to_F2xX(GEN x, long v)
GEN FlxX_to_F2xX(GEN x)
GEN F2xX_to_ZXX(GEN B)
GEN F2xX_renormalize(GEN x, long lx)
long F2xY_degrees(GEN P) return the degree of P with respect to the secondary variable.
GEN pol1_F2xX(long v, long sv)
GEN polx_F2xX(long v, long sv)
GEN F2xX_add(GEN x, GEN y)
GEN F2xX_F2x_add(GEN x, GEN y)
GEN F2xX_F2x_mul(GEN x, GEN y)
GEN F2xX_deriv(GEN P) returns the derivative of P with respect to the main variable.
GEN Kronecker_to_F2xqX(GEN z, GEN T)
GEN F2xX_to_Kronecker(GEN z, GEN T)
GEN F2xY_F2xq_evalx(GEN x, GEN y, GEN T) as FpXY_FpXQ_evalx.
GEN F2xY_F2xqV_evalx(GEN x, GEN V, GEN T) as FpXY_FpXQV_evalx.

7.3.27 F2xXV/F2xXC.. See FpXXV operations.

GEN F1xXC_to_F2xXC(GEN B)

GEN F2xXC_to_ZXXC(GEN B)

7.3.28 F2xqX.. See F1xqX operations.

7.3.28.1 Preconditioned reduction.

For faster reduction, the modulus S can be replaced by an extended modulus, which is an F2xqXT, in all F2xqXQ-classes functions, and in F2xqX.rem and F2xqX.divrem.

GEN F2xqX_get_red(GEN S, GEN T) returns the extended modulus eS .

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get_F2xqX_mod(GEN eS) returns the underlying modulus S .

GEN get_F2xqX_var(GEN eS) returns the variable number of the modulus.

GEN get_F2xqX_degree(GEN eS) returns the degree of the modulus.

7.3.28.2 basic functions.

GEN random_F2xqX(long d, long v, GEN T, ulong p) returns a random F2xqX in variable v , of degree less than d .

GEN F2xqX_red(GEN z, GEN T)

GEN F2xqX_normalize(GEN z, GEN T)

GEN F2xqX_F2xq_mul(GEN P, GEN U, GEN T)

GEN F2xqX_F2xq_mul_to_monic(GEN P, GEN U, GEN T)

GEN F2xqX_mul(GEN x, GEN y, GEN T)

GEN F2xqX_sqr(GEN x, GEN T)

GEN F2xqX_powu(GEN x, ulong n, GEN T)

GEN F2xqX_rem(GEN x, GEN y, GEN T)

GEN F2xqX_div(GEN x, GEN y, GEN T)

GEN F2xqX_divrem(GEN x, GEN y, GEN T, GEN *pr)

GEN F2xqXQ_inv(GEN x, GEN S, GEN T)

GEN F2xqXQ_invsafe(GEN x, GEN S, GEN T)

GEN F2xqX_invBarrett(GEN T, GEN Q)

GEN F2xqX_extgcd(GEN x, GEN y, GEN T, GEN *ptu, GEN *ptv)

GEN F2xqX_gcd(GEN x, GEN y, GEN T)

long F2xqX_ispower(GEN f, ulong k, GEN T, GEN *pt)

GEN F2xqX_F2xqXQ_eval(GEN Q, GEN x, GEN S, GEN T) as FpX.FpXQ_eval.

GEN F2xqX_F2xqXQV_eval(GEN P, GEN V, GEN S, GEN T) as FpX.FpXQV_eval.

GEN F2xqX_roots(GEN f, GEN T) return the roots of f in $\mathbf{F}_2[X]/(T)$. Assumes T irreducible in $\mathbf{F}_2[X]$.

GEN F2xqX_factor(GEN f, GEN T) return the factorization of f over $\mathbf{F}_2[X]/(T)$. Assumes T irreducible in $\mathbf{F}_2[X]$.

GEN F2xqX_factor_squarefree(GEN f, GEN T) as FlxqX_factor_squarefree.

GEN F2xqX_ddf(GEN f, GEN T) as FpX_ddf.

GEN F2xqX_degfact(GEN f, GEN T) as FpX_degfact.

7.3.29 F2xqXQ.. See FlxqXQ operations.

GEN FlxqXQ_inv(GEN x, GEN S, GEN T)

GEN FlxqXQ_invsafe(GEN x, GEN S, GEN T)

GEN F2xqXQ_mul(GEN x, GEN y, GEN S, GEN T)

GEN F2xqXQ_sqr(GEN x, GEN S, GEN T)

GEN F2xqXQ_pow(GEN x, GEN n, GEN S, GEN T)

GEN F2xqXQ_powers(GEN x, long n, GEN S, GEN T)

GEN F2xqXQ_autpow(GEN a, long n, GEN S, GEN T) as FpXQXQ_autpow

GEN F2xqXQ_auttrace(GEN a, long n, GEN S, GEN T). Let σ be the automorphism defined by $\sigma(X) = a[1] \pmod{T(X)}$ and $\sigma(Y) = a[2] \pmod{S(X,Y),T(X)}$; returns the vector $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$ where $b = a[3]$.

GEN F2xqXQV_red(GEN x, GEN S, GEN T)

7.3.30 Functions returning objects with t_INTMOD coefficients.

Those functions are mostly needed for interface reasons: t_INTMODs should not be used in library mode since the modular kernel is more flexible and more efficient, but GP users do not have access to the modular kernel. We document them for completeness:

GEN Fp_to_mod(GEN z, GEN p), z a t_INT. Returns $z * \text{Mod}(1,p)$, normalized. Hence the returned value is a t_INTMOD.

GEN FpX_to_mod(GEN z, GEN p), z a ZX. Returns $z * \text{Mod}(1,p)$, normalized. Hence the returned value has t_INTMOD coefficients.

GEN FpC_to_mod(GEN z, GEN p), z a ZC. Returns $\text{Col}(z) * \text{Mod}(1,p)$, a t_COL with t_INTMOD coefficients.

GEN FpV_to_mod(GEN z, GEN p), z a ZV. Returns $\text{Vec}(z) * \text{Mod}(1,p)$, a t_VEC with t_INTMOD coefficients.

GEN FpVV_to_mod(GEN z, GEN p), z a ZVV. Returns $\text{Vec}(z) * \text{Mod}(1,p)$, a t_VEC of t_VEC with t_INTMOD coefficients.

GEN FpM_to_mod(GEN z, GEN p), z a ZM. Returns $z * \text{Mod}(1,p)$, with t_INTMOD coefficients.

GEN F2c_to_mod(GEN x)

GEN F2m_to_mod(GEN x)

GEN Flc_to_mod(GEN z)

GEN Flm_to_mod(GEN z)

GEN FqM_to_mod(GEN z, GEN T, GEN p)

GEN FpXQC_to_mod(GEN V, GEN T, GEN p) V being a vector of FpXQ, converts each entry to a t_POLMOD with t_INTMOD coefficients, and return a t_COL.

GEN QXQV_to_mod(GEN V, GEN T) V a vector of QXQ, which are lifted representatives of elements of $\mathbf{Q}[X]/(T)$ (number field elements in most applications) and T is in $\mathbf{Z}[X]$. Return a vector where all non-rational entries are converted to t_POLMOD modulo T ; no reduction mod T is attempted: the representatives should be already reduced. Used to normalize the output of nroots.

GEN QXQX_to_mod_shallow(GEN P, GEN T) P a polynomial with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

GEN QXQC_to_mod_shallow(GEN V, GEN T) V a vector with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

GEN QXQM_to_mod_shallow(GEN M, GEN T) M a matrix with QXQ coefficients; replace them by mkpolmod(.,T). Shallow function.

GEN QXQXV_to_mod(GEN V, GEN T) V a vector of polynomials whose coefficients are QXQ. Analogous to QXQV_to_mod. Used to normalize the output of nffactor.

The following functions are obsolete and should not be used: they receive a polynomial with arbitrary coefficients, apply a conversion function to map them to a finite field, a function from the modular kernel, then *_to_mod:

GEN rootmod(GEN f, GEN p), applies FpX_roots.

GEN rootmod2(GEN f, GEN p), (now) identical to rootmod.

GEN rootmod0(GEN f, GEN p, long flag), calls either rootmod or rootmod2 depending on flag.

GEN factmod(GEN f, GEN p) applies *_factor.

GEN simplefactmod(GEN f, GEN p) applies *_degfact.

7.3.31 Slow Chinese remainder theorem over \mathbf{Z} . The routines in this section have quadratic time complexity with respect to the input size; see the routines in the next two sections for quasi-linear time variants.

GEN Z_chinese(GEN a, GEN b, GEN A, GEN B) returns the integer in $[0, \text{lcm}(A, B)[$ congruent to $a \bmod A$ and $b \bmod B$, assuming it exists; in other words, that a and b are congruent mod $\text{gcd}(A, B)$.

GEN Z_chinese_all(GEN a, GEN b, GEN A, GEN B, GEN *pC) as Z_chinese, setting *pC to the lcm of A and B .

GEN Z_chinese_coprime(GEN a, GEN b, GEN A, GEN B, GEN C), as Z_chinese, assuming that $\text{gcd}(A, B) = 1$ and that $C = \text{lcm}(A, B) = AB$.

ulong u_chinese_coprime(ulong a, ulong b, ulong A, ulong B, ulong C), as Z_chinese_coprime for ulong inputs and output.

void Z_chinese_pre(GEN A, GEN B, GEN *pC, GEN *pU, GEN *pd) initializes chinese remainder computations modulo A and B . Sets *pC to $\text{lcm}(A, B)$, *pd to $\text{gcd}(A, B)$, *pU to an integer

congruent to 0 mod (A/d) and 1 mod (B/d) . It is allowed to set `pd = NULL`, in which case, d is still computed, but not saved.

`GEN Z_chinese_post(GEN a, GEN b, GEN C, GEN U, GEN d)` returns the solution to the chinese remainder problem x congruent to a mod A and b mod B , where C, U, d were set in `Z_chinese_pre`. If d is `NULL`, assume the problem has a solution. Otherwise, return `NULL` if it has no solution.

The following pair of functions is used in homomorphic imaging schemes, when reconstructing an integer from its images modulo pairwise coprime integers. The idea is as follows: we want to discover an integer H which satisfies $|H| < B$ for some known bound B ; we are given pairs (H_p, p) with H congruent to H_p mod p and all p pairwise coprime.

Given H congruent to H_p modulo a number of p , whose product is q , and a new pair (H_p, p) , p coprime to q , the following incremental functions use the chinese remainder theorem (CRT) to find a new H , congruent to the preceding one modulo q , but also to H_p modulo p . It is defined uniquely modulo qp , and we choose the centered representative. When P is larger than $2B$, we have $H = H$, but of course, the value of H may stabilize sooner. In many applications it is possible to directly check that such a partial result is correct.

`GEN Z_init_CRT(ulong Hp, ulong p)` given a `Fl Hp` in $[0, p-1]$, returns the centered representative H congruent to H_p modulo p .

`int Z_incremental_CRT(GEN *H, ulong Hp, GEN *q, ulong p)` given a `t_INT *H`, centered modulo $*q$, a new pair (H_p, p) with p coprime to q , this function updates $*H$ so that it also becomes congruent to (H_p, p) , and $*q$ to the product $qp = p \cdot *q$. It returns 1 if the new value is equal to the old one, and 0 otherwise.

`GEN chinese1_coprime_Z(GEN v)` an alternative divide-and-conquer implementation: v is a vector of `t_INTMOD` with pairwise coprime moduli. Return the `t_INTMOD` solving the corresponding chinese remainder problem. This is a streamlined version of

`GEN chinese1(GEN v)`, which solves a general chinese remainder problem (not necessarily over \mathbf{Z} , moduli not assumed coprime).

As above, for H a `ZM`: we assume that H and all H_p have dimension > 0 . The original $*H$ is destroyed.

`GEN ZM_init_CRT(GEN Hp, ulong p)`

`int ZM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above for H a `ZX`: note that the degree may increase or decrease. The original $*H$ is destroyed.

`GEN ZX_init_CRT(GEN Hp, ulong p, long v)`

`int ZX_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above, for H a matrix whose coefficient are `ZX`. The original $*H$ is destroyed. The entries of H are not normalized, use `ZX_renormalize` for this.

`GEN ZXM_init_CRT(GEN Hp, long deg, ulong p)` where `deg` is the maximal degree of all the H_p

`int ZXM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

7.3.32 Fast remainders.

The routines in these section are asymptotically fast (quasi-linear time in the input size).

GEN Z_ZV_mod(GEN A, GEN P) given a `t_INT` A and a vector P of positive pairwise coprime integers of length $n \geq 1$, return a vector B of the same length such that $B[i] = A \pmod{P[i]}$ and $0 \leq B[i] < P[i]$ for all $1 \leq i \leq n$. The vector P may be a `t_VEC` or a `t_VECSMALL` (treated as `ulongs`) and B has the same type as P .

GEN Z_nv_mod(GEN A, GEN P) given a `t_INT` A and a `t_VECSMALL` P of positive pairwise coprime integers of length $n \geq 1$, return a `t_VECSMALL` B of the same length such that $B[i] = A \pmod{P[i]}$ and $0 \leq B[i] < P[i]$ for all $1 \leq i \leq n$. The entries of P and B are treated as `ulongs`.

The following low level functions allow precomputations:

GEN ZV_producttree(GEN P) where P is a vector of integers (or `t_VECSMALL`) of length $n \geq 1$, return the vector of `t_VECS` $[f(P), f^2(P), \dots, f^k(P)]$ where f is the transformation $[p_1, p_2, \dots, p_m] \mapsto [p_1 p_2, p_3 p_4, \dots, p_{m-1} p_m]$ if m is even and $[p_1 p_2, p_3 p_4, \dots, p_{m-2} p_{m-1}, p_m]$ if m is odd, and $k = O(\log m)$ is minimal so that $f^k(P)$ has length 1; in other words, $f^k(P) = [p_1 p_2 \dots p_m]$.

GEN Z_ZV_mod_tree(GEN A, GEN P, GEN T) as `Z_ZV_mod` where T is the tree `ZV_producttree(P)`.

GEN ZV_nv_mod_tree(GEN A, GEN P, GEN T) A being a `ZV` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flv` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

GEN ZM_nv_mod_tree(GEN A, GEN P, GEN T) A being a `ZM` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flm` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

GEN ZX_nv_mod_tree(GEN A, GEN P, GEN T) A being a `ZX` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `Flx` polynomials $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

GEN ZXC_nv_mod_tree(GEN A, GEN P, GEN T) A being a `ZXC` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxC` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

GEN ZXM_nv_mod_tree(GEN A, GEN P, GEN T) A being a `ZXM` and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxM` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is the tree `ZV_producttree(P)`.

GEN ZXX_nv_mod_tree(GEN A, GEN P, GEN T, long v) A being a `ZXX`, and P a `t_VECSMALL` of length $n \geq 1$, the elements of P being pairwise coprime, return the vector of `FlxX` $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$, where T is assumed to be the tree created by `ZV_producttree(P)`.

7.3.33 Fast Chinese remainder theorem over \mathbf{Z} . The routines in these section are asymptotically fast (quasi-linear time in the input size) and should be used whenever the moduli are known from the start.

The simplest function is

`GEN ZV_chinese(GEN A, GEN P, GEN *pM)` let P be a vector of positive pairwise coprime integers, let A be a vector of integers of the same length $n \geq 1$ such that $0 \leq A[i] < P[i]$ for all i , and let M be the product of the elements of P . Returns the integer in $[0, M[$ congruent to $A[i] \bmod P[i]$ for all $1 \leq i \leq n$. If `pM` is not NULL, set `*pM` to M . We also allow `t_VECSMALLs` for A and P (seen as vectors of unsigned integers).

`GEN ZV_chinese_center(GEN A, GEN P, GEN *pM)` As `ZV_chinese` but return integers in $[-M/2, M/2[$ instead.

The following functions allow to solve many Chinese remainder problems simultaneously, for a given set of moduli:

`GEN nxV_chinese_center(GEN A, GEN P, GEN *pt_mod)` where A is a vector of `nx` and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P , returns the `t_POL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i] \bmod P[i]$ for all $1 \leq i \leq n$. If `pt_mod` is not NULL, set `*pt_mod` to M .

`GEN ncV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `VECSMALLs` (seen as vectors of unsigned integers) and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P , returns the `t_COL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i] \bmod P[i]$ for all $1 \leq i \leq n$. If `pM` is not NULL, set `*pt_mod` to M .

`GEN nmV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `MATSMALLs` (seen as matrices of unsigned integers) and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P , returns the matrix whose entries are integers in $[-M/2, M/2[$ congruent to $A[i] \bmod P[i]$ for all $1 \leq i \leq n$. If `pM` is not NULL, set `*pM` to M . N.B.: this function uses the parallel GP interface.

`GEN nxCV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `nxCs` and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P , returns the `t_COL` whose entries are integers in $[-M/2, M/2[$ congruent to $A[i] \bmod P[i]$ for all $1 \leq i \leq n$. If `pM` is not NULL, set `*pt_mod` to M .

`GEN nxMV_chinese_center(GEN A, GEN P, GEN *pM)` where A is a vector of `nxMs` and P a `t_VECSMALL` of the same length $n \geq 1$, the elements of P being pairwise coprime, and M being the product of the elements of P , returns the matrix whose entries are integers in $[-M/2, M/2[$ congruent to $A[i] \bmod P[i]$ for all $1 \leq i \leq n$. If `pM` is not NULL, set `*pM` to M . N.B.: this function uses the parallel GP interface.

The other routines allow for various precomputations :

`GEN ZV_chinesetree(GEN P, GEN T)` given P a vector of integers (or `t_VECSMALL`) and a product tree T from `ZV_producttree(P)` for the same P , return a “chinese remainder tree” R , preconditioning the solution of Chinese remainder problems modulo the $P[i]$.

`GEN ZV_chinese_tree(GEN A, GEN P, GEN T, GEN R)` return `ZV_chinese(A, P, NULL)`, where T is created by `ZV_producttree(P)` and R by `ZV_chinesetree(P, T)`.

GEN nmV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R) as nmV_chinese_center where T is assumed to be the tree created by ZV_producttree(P) and R by ZV_chinesetree(P, T).

GEN nxV_chinese_center_tree(GEN A, GEN P, GEN T, GEN R) as nxV_chinese_center where T is assumed to be the tree created by ZV_producttree(P) and R by ZV_chinesetree(P, T).

7.3.34 Rational reconstruction.

int Fp_ratlift(GEN x, GEN m, GEN amax, GEN bmax, GEN *a, GEN *b). Assuming that $0 \leq x < m$, $\text{amax} \geq 0$, and $\text{bmax} > 0$ are t_INTs, and that $2\text{amaxbmax} < m$, attempts to recognize x as a rational a/b , i.e. to find t_INTs a and b such that

- $a \equiv bx$ modulo m ,
- $|a| \leq \text{amax}$, $0 < b \leq \text{bmax}$,
- $\text{gcd}(m, b) = \text{gcd}(a, b)$.

If unsuccessful, the routine returns 0 and leaves a, b unchanged; otherwise it returns 1 and sets a and b .

In almost all applications, we actually know that a solution exists, as well as a non-zero multiple B of b , and $m = p^\ell$ is a prime power, for a prime p chosen coprime to B hence to b . Under the single assumption $\text{gcd}(m, b) = 1$, if a solution a, b exists satisfying the three conditions above, then it is unique.

GEN FpM_ratlift(GEN M, GEN m, GEN amax, GEN bmax, GEN denom) given an FpM modulo m with reduced or Fp_center-ed entries, reconstructs a matrix with rational coefficients by applying Fp_ratlift to all entries. Assume that all preconditions for Fp_ratlift are satisfied, as well $\text{gcd}(m, b) = 1$ (so that the solution is unique if it exists). Return NULL if the reconstruction fails, and the rational matrix otherwise. If denom is not NULL check further that all denominators divide denom .

The function is not stack clean if one of the coefficients of M is negative (centered residues), but still suitable for gerepileupto.

GEN FpX_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as FpM_ratlift, where P is an FpX.

GEN FpC_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as FpM_ratlift, where P is an FpC.

7.3.35 Zp.

GEN Zp_sqrt(GEN b, GEN p, long e) b and p being t_INTs, with p a prime (possibly 2), returns a t_INT a such that $a^2 \equiv b \pmod{p^e}$.

GEN Z2_sqrt(GEN b, long e) b being a t_INTs returns a t_INT a such that $a^2 \equiv b \pmod{2^e}$.

GEN Zp_sqrtlift(GEN b, GEN a, GEN p, long e) let a, b, p be t_INTs, with $p > 1$ odd, such that $a^2 \equiv b \pmod{p}$. Returns a t_INT A such that $A^2 \equiv b \pmod{p^e}$. Special case of Zp_sqrtnlift.

GEN Zp_sqrtnlift(GEN b, GEN n, GEN a, GEN p, long e) let a, b, n, p be t_INTs, with $n, p > 1$, and p coprime to n , such that $a^n \equiv b \pmod{p}$. Returns a t_INT A such that $A^n \equiv b \pmod{p^e}$. Special case of ZpX_liftroot.

GEN `Zp_teachmuller`(GEN `x`, GEN `p`, long `e`, GEN `pe`) for p an odd prime, x a `t_INT` coprime to p , and $pe = p^e$, returns the $(p - 1)$ -th root of 1 congruent to x modulo p , modulo p^e . For convenience, $p = 2$ is also allowed and we return 1 (x is 1 mod 4) or $2^e - 1$ (x is 3 mod 4).

GEN `teichmullerinit`(long `p`, long `n`) returns the values of `Zp_teachmuller` at all $x = 1, \dots, p - 1$.

7.3.36 `ZpX`.

GEN `ZpX_roots`(GEN `f`, GEN `p`, long `e`) f a `ZX` with leading term prime to p , and without multiple roots mod p . Return a vector of `t_INTs` which are the roots of f mod p^e .

GEN `ZpX_liftroot`(GEN `f`, GEN `a`, GEN `p`, long `e`) f a `ZX` with leading term prime to p , and a a root mod p such that $v_p(f'(a)) = 0$. Return a `t_INT` which is the root of f mod p^e congruent to a mod p .

GEN `ZX_Zp_root`(GEN `f`, GEN `a`, GEN `p`, long `e`) same as `ZpX_liftroot` without the assumption $v_p(f'(a)) = 0$. Return a `t_VEC` of `t_INTs`, which are the p -adic roots of f congruent to a mod p (given modulo p^e).

GEN `ZpX_liftroots`(GEN `f`, GEN `S`, GEN `p`, long `e`) f a `ZX` with leading term prime to p , and S a vector of simple roots mod p . Return a vector of `t_INTs` which are the root of f mod p^e congruent to the $S[i]$ mod p .

GEN `ZpX_liftfact`(GEN `A`, GEN `B`, GEN `pe`, GEN `p`, long `e`) is the routine underlying `pol-hensellift`. Here, p is prime defines a finite field \mathbf{F}_p . A is a polynomial in $\mathbf{Z}[X]$, whose leading coefficient is non-zero in \mathbf{F}_q . B is a vector of monic `FpX`, pairwise coprime in $\mathbf{F}_p[X]$, whose product is congruent to $A/\text{lc}(A)$ in $\mathbf{F}_p[X]$. Lifts the elements of B mod $pe = p^e$.

GEN `ZpX_Frobenius`(GEN `T`, GEN `p`, ulong `e`) returns the p -adic lift of the Frobenius automorphism of $\mathbf{F}_p[X]/(T)$ to precision e .

long `ZpX_disc_val`(GEN `f`, GEN `p`) returns the valuation at p of the discriminant of f . Assume that f is a monic *separable* `ZX` and that p is a prime number. Proceeds by dynamically increasing the p -adic accuracy; infinite loop if the discriminant of f is 0.

long `ZpX_resultant_val`(GEN `f`, GEN `g`, GEN `p`, long `M`) returns the valuation at p of $\text{Res}(f, g)$. Assume f, g are both `ZX`, and that p is a prime number coprime to the leading coefficient of f . Proceeds by dynamically increasing the p -adic accuracy. To avoid an infinite loop when the resultant is 0, we return M if the Sylvester matrix mod p^M still does not have maximal rank.

GEN `ZpX_gcd`(GEN `f`, GEN `g`, GEN `p`, GEN `pm`) f a monic `ZX`, g a `ZX`, $pm = p^m$ a prime power. There is a unique integer $r \geq 0$ and a monic $h \in \mathbf{Q}_p[X]$ such that

$$p^r h \mathbf{Z}_p[X] + p^m \mathbf{Z}_p[X] = f \mathbf{Z}_p[X] + g \mathbf{Z}_p[X] + p^m \mathbf{Z}_p[X].$$

Return the 0 polynomial if $r \geq m$ and a monic $h \in \mathbf{Z}[1/p][X]$ otherwise (whose valuation at p is $> -m$).

GEN `ZpX_reduced_resultant`(GEN `f`, GEN `g`, GEN `p`, GEN `pm`) f a monic `ZX`, g a `ZX`, $pm = p^m$ a prime power. The p -adic *reduced resultant* of f and g is 0 if f, g not coprime in $\mathbf{Z}_p[X]$, and otherwise the generator of the form p^d of

$$(f \mathbf{Z}_p[X] + g \mathbf{Z}_p[X]) \cap \mathbf{Z}_p.$$

Return the reduced resultant modulo p^m .

`GEN ZpX_reduced_resultant_fast(GEN f, GEN g, GEN p, long M)` f a monic ZX, g a ZX, p a prime. Returns the p -adic reduced resultant of f and g modulo p^M . This function computes resultants for a sequence of increasing p -adic accuracies (up to M p -adic digits), returning as soon as it obtains a non-zero result. It is very inefficient when the resultant is 0, but otherwise usually more efficient than computations using a priori bounds.

`GEN ZpX_mononic_factor(GEN f, GEN p, long M)` f a monic ZX, p a primer, return the p -adic factorization of f , modulo p^M . This is the underlying low-level recursive function behind `factorpadic` (using a combination of Round 4 factorization and Hensel lifting); the factors are not sorted and the function is not `gerepile-clean`.

7.3.37 ZpXQ.

`GEN ZpXQ_invlift(GEN b, GEN a, GEN T, GEN p, long e)` let p be a prime `t_INT`, a be a `FpXQ` (modulo (p, T)) and b a `ZpXQ` such that $ab \equiv 1 \pmod{(p, T)}$. Returns an `ZpXQ` A such that $A \equiv a \pmod{p}$ and $Ab \equiv 1 \pmod{(p^e, T)}$.

`GEN ZpXQ_inv(GEN b, GEN T, GEN p, long e)` let p be a prime `t_INT` and b be a `FpXQ` (modulo T, p^e). Returns an `FpXQ` A such that $Ab \equiv 1 \pmod{(p^e, T)}$.

`GEN ZpXQ_div(GEN a, GEN b, GEN T, GEN q, GEN p, long e)` let p be a prime `t_INT` and a and b be a `FpXQ` (modulo T, p^e). Returns an `FpXQ` c such that $cb \equiv a \pmod{(p^e, T)}$. The parameter q must be equal to p^e .

`GEN ZpXQ_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e)` let n, p be `t_INTs`, with $n, p > 1$ and p coprime to n , and a, b be `FpXQs` (modulo T) such that $a^n \equiv b \pmod{(p, T)}$. Returns an `Fq` A such that $A^n \equiv b \pmod{(p^e, T)}$.

`GEN ZpXQ_sqrt(GEN b, GEN T, GEN p, long e)` let p being a odd prime and b be a `FpXQ` (modulo T, p^e), returns a such that $a^2 \equiv b \pmod{(p^e, T)}$.

`GEN ZpX_ZpXQ_liftroot(GEN f, GEN a, GEN T, GEN p, long e)` as `ZpXQX_liftroot`, but f is a polynomial in $\mathbf{Z}[X]$.

`GEN ZpX_ZpXQ_liftroot_early(GEN f, GEN a, GEN T, GEN p, long e, void *E, int early(void *E, GEN x, GEN q))` as `ZpX_ZpXQ_liftroot` with early abort: the function `early(E, x, q)` will be called with x is a root of f modulo $q = p^n$ for some n . If `early` returns a non-zero value, the function returns x immediately.

`GEN ZpXQ_log(GEN a, GEN T, GEN p, long e)` T being a `ZpX` irreducible modulo p , return the logarithm of a in $\mathbf{Z}_p[X]/(T)$ to precision e , assuming that $a \equiv 1 \pmod{p\mathbf{Z}_p[X]}$ if p odd or $a \equiv 1 \pmod{4\mathbf{Z}_2[X]}$ if $p = 2$.

7.3.38 Zq.

`GEN Zq_sqrtnlift(GEN b, GEN n, GEN a, GEN T, GEN p, long e)`

7.3.39 ZpXQM.

`GEN ZpXQM_prodFrobenius(GEN M, GEN T, GEN p, long e)` returns the product of matrices $M\sigma(M)\sigma^2(M)\dots\sigma^{n-1}(M)$ to precision e where σ is the lift of the Frobenius automorphism over $\mathbf{Z}_p[X]/(T)$ and n is the degree of T .

7.3.40 ZpXQX.

GEN ZpXQX_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e) is the routine underlying polhensellift. Here, p is prime, $T(Y)$ defines a finite field \mathbf{F}_q . A is a polynomial in $\mathbf{Z}[X, Y]$, whose leading coefficient is non-zero in \mathbf{F}_q . B is a vector of monic or FqX, pairwise coprime in $\mathbf{F}_q[X]$, whose product is congruent to $A/\text{lc}(A)$ in $\mathbf{F}_q[X]$. Lifts the elements of $B \bmod pe = p^e$, such that the congruence now holds mod (T, p^e) .

GEN ZpXQX_liftroot(GEN f, GEN a, GEN T, GEN p, long e) as ZpX_liftroot, but f is now a polynomial in $\mathbf{Z}[X, Y]$ and lift the root a in the unramified extension of \mathbf{Q}_p with residue field $\mathbf{F}_p[Y]/(T)$, assuming $v_p(f(a)) > 0$ and $v_p(f'(a)) = 0$.

GEN ZpXQX_liftroot_vald(GEN f, GEN a, long v, GEN T, GEN p, long e) returns the foofs of f as ZpXQX_liftroot, where v is the valuation of the content of f' and it is required that $v_p(f(a)) > v$ and $v_p(f'(a)) = v$.

GEN ZpXQX_roots(GEN F, GEN T, GEN p, long e)

GEN ZpXQX_divrem(GEN x, GEN Sp, GEN T, GEN q, GEN p, long e, GEN *pr) as FpXQX_divrem. The parameter q must be equal to p^e .

GEN ZpXQX_digits(GEN x, GEN B, GEN T, GEN q, GEN p, long e) As FpXQX_digits. The parameter q must be equal to p^e .

7.3.41 ZqX.

GEN ZqX_roots(GEN F, GEN T, GEN p, long e)

GEN ZqX_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e)

GEN ZqX_liftroot(GEN f, GEN a, GEN T, GEN p, long e)

7.3.42 Other p -adic functions.

GEN ZpM_echelon(GEN M, long early_abort, GEN p, GEN pm) given a ZM M , a prime p and $pm = p^m$, returns an echelon form E for $M \bmod p^m$. I.e. there exist a square integral matrix U with $\det U$ coprime to p such that $E = MU$ modulo p^m . If **early_abort** is non-zero, return NULL as soon as one pivot in the echelon form is divisible by p^m . The echelon form is an upper triangular HNF, we do not waste time to reduce it to Gauss-Jordan form.

GEN zlm_echelon(GEN M, long early_abort, ulong p, ulong pm) variant of ZpM_echelon, for a Zlm M .

GEN Zlm_gauss(GEN a, GEN b, ulong p, long e, GEN C) as gauss with the following peculiarities: a and b are ZM, such that a is invertible modulo p . Optional C is an Flm that is an inverse of $a \bmod p$ or NULL. Return the matrix x such that $ax = b \bmod p^e$ and all elements of x are in $[0, p^e - 1]$. For efficiency, it is better to reduce a and $b \bmod p^e$ first.

GEN padic_to_Q(GEN x) truncate the t_PADIC to a t_INT or t_FRAC.

GEN padic_to_Q_shallow(GEN x) shallow version of padic_to_Q

GEN QpV_to_QV(GEN v) apply padic_to_Q_shallow

long padicprec(GEN x, GEN p) returns the absolute p -adic precision of the object x , by definition the minimum precision of the components of x . For a non-zero t_PADIC, this returns $\text{valp}(x) + \text{precp}(x)$.

`long padicprec_relative(GEN x)` returns the relative p -adic precision of the `t_INT`, `t_FRAC`, or `t_PADIC` x (minimum precision of the components of x for `t_POL` or vector/matrices). For a `t_PADIC`, this returns `precp(x)` if $x \neq 0$, and 0 for $x = 0$.

7.3.42.1 low-level.

The following technical function returns an optimal sequence of p -adic accuracies, for a given target accuracy:

`ulong quadratic_prec_mask(long n)` we want to reach accuracy $n \geq 1$, starting from accuracy 1, using a quadratically convergent, self-correcting, algorithm; in other words, from inputs correct to accuracy l one iteration outputs a result correct to accuracy $2l$. For instance, to reach $n = 9$, we want to use accuracies $[1, 2, 3, 5, 9]$ instead of $[1, 2, 4, 8, 9]$. The idea is to essentially double the accuracy at each step, and not overshoot in the end.

Let $a_0 = 1, a_1 = 2, \dots, a_k = n$, be the desired sequence of accuracies. To obtain it, we work backwards and set

$$a_k = n, \quad a_{i-1} = (a_i + 1) \setminus 2.$$

This is in essence what the function returns. But we do not want to store the a_i explicitly, even as a `t_VECSMALL`, since this would leave an object on the stack. Instead, we store a_i implicitly in a bitmask `MASK`: let $a_0 = 1$, if the i -th bit of the mask is set, set $a_{i+1} = 2a_i - 1$, and $2a_i$ otherwise; in short the bits indicate the places where we do something special and do not quite double the accuracy (which would be the straightforward thing to do).

In fact, to avoid returning separately the mask and the sequence length $k + 1$, the function returns `MASK + 2k+1`, so the highest bit of the mask indicates the length of the sequence, and the following ones give an algorithm to obtain the accuracies. This is much simpler than it sounds, here is what it looks like in practice:

```
ulong mask = quadratic_prec_mask(n);
long l = 1;
while (mask > 1) {
    /* here, the result is known to accuracy l */
    l = 2*l; if (mask & 1) l--; /* new accuracy l for the iteration */
    mask >>= 1; /* pop low order bit */
    /* ... lift to the new accuracy ... */
}
/* we are done. At this point l = n */
```

We just pop the bits in `mask` starting from the low order bits, stop when `mask` is 1 (that last bit corresponds to the 2^{k+1} that we added to the mask proper). Note that there is nothing specific to Hensel lifts in that function: it would work equally well for an Archimedean Newton iteration.

Note that in practice, we rather use an infinite loop, and insert an

```
if (mask == 1) break;
```

in the middle of the loop: the loop body usually includes preparations for the next iterations (e.g. lifting Bezout coefficients in a quadratic Hensel lift), which are costly and useless in the *last* iteration.

7.3.43 Conversions involving single precision objects.

7.3.43.1 To single precision.

ulong Rg_to_F1(GEN z, ulong p), z which can be mapped to $\mathbf{Z}/p\mathbf{Z}$: a t_INT, a t_INTMOD whose modulus is divisible by p, a t_FRAC whose denominator is coprime to p, or a t_PADIC with underlying prime ℓ satisfying $p = \ell^n$ for some n (less than the accuracy of the input). Returns lift(z * Mod(1,p)), normalized, as an Fl.

ulong Rg_to_F2(GEN z), as Rg_to_F1 for $p = 2$.

ulong padic_to_F1(GEN x, ulong p) special case of Rg_to_F1, for a x a t_PADIC.

GEN RgX_to_F2x(GEN x), x a t_POL, returns the F2x obtained by applying Rg_to_F1 coefficientwise.

GEN RgX_to_Flx(GEN x, ulong p), x a t_POL, returns the Flx obtained by applying Rg_to_F1 coefficientwise.

GEN Rg_to_F2xq(GEN z, GEN T), z a GEN which can be mapped to $\mathbf{F}_2[X]/(T)$: anything Rg_to_F1 can be applied to, a t_POL to which RgX_to_F2x can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a F2x), a suitable t_RFRAC. Returns z as an F2xq, normalized.

GEN Rg_to_Flxq(GEN z, GEN T, ulong p), z a GEN which can be mapped to $\mathbf{F}_p[X]/(T)$: anything Rg_to_F1 can be applied to, a t_POL to which RgX_to_Flx can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a Flx), a suitable t_RFRAC. Returns z as an Flxq, normalized.

GEN RgX_to_FlxqX(GEN z, GEN T, ulong p), z a GEN which can be mapped to $\mathbf{F}_p[x]/(T)[X]$: anything Rg_to_Flxq can be applied to, a t_POL to which RgX_to_Flx can be applied to, a t_POLMOD whose modulus is divisible by T (once mapped to a Flx), a suitable t_RFRAC. Returns z as an FlxqX, normalized.

GEN ZX_to_Flx(GEN x, ulong p) reduce ZX x modulo p (yielding an Flx). Faster than RgX_to_Flx.

GEN ZV_to_Flv(GEN x, ulong p) reduce ZV x modulo p (yielding an Flv).

GEN ZXV_to_FlxV(GEN v, ulong p), as ZX_to_Flx, repeatedly called on the vector's coefficients.

GEN ZXT_to_FlxT(GEN v, ulong p), as ZX_to_Flx, repeatedly called on the tree leaves.

GEN ZXX_to_FlxX(GEN B, ulong p, long v), as ZX_to_Flx, repeatedly called on the polynomial's coefficients.

GEN zxX_to_FlxX(GEN z, ulong p) as zx_to_Flx, repeatedly called on the polynomial's coefficients.

GEN ZXXV_to_FlxXV(GEN V, ulong p, long v), as ZXX_to_FlxX, repeatedly called on the vector's coefficients.

GEN ZXXT_to_FlxXT(GEN V, ulong p, long v), as ZXX_to_FlxX, repeatedly called on the tree leaves.

GEN RgV_to_Flv(GEN x, ulong p) reduce the t_VEC/t_COL x modulo p, yielding a t_VECSMALL.

GEN RgM_to_Flm(GEN x, ulong p) reduce the t_MAT x modulo p.

GEN ZM_to_Flm(GEN x, ulong p) reduce ZM x modulo p (yielding an Flm).

GEN ZV_to_zv(GEN z), converts coefficients using itos

GEN ZV_to_nv(GEN z), converts coefficients using itou

GEN ZM_to_zm(GEN z), converts coefficients using itos

GEN FqC_to_FlxC(GEN x, GEN T, GEN p), converts coefficients in Fq to coefficient in Flx, result being a column vector.

GEN FqV_to_FlxV(GEN x, GEN T, GEN p), converts coefficients in Fq to coefficient in Flx, result being a line vector.

GEN FqM_to_FlxM(GEN x, GEN T, GEN p), converts coefficients in Fq to coefficient in Flx.

7.3.43.2 From single precision.

GEN Flx_to_ZX(GEN z), converts to ZX (t_POL of non-negative t_INTs in this case)

GEN Flx_to_FlxX(GEN z), converts to FlxX (t_POL of constant Flx in this case).

GEN Flx_to_ZX_inplace(GEN z), same as Flx_to_ZX, in place (z is destroyed).

GEN FlxX_to_ZXX(GEN B), converts an FlxX to a polynomial with ZX or t_INT coefficients (repeated calls to Flx_to_ZX).

GEN FlxXC_to_ZXXC(GEN B), converts an FlxXC to a t_COL with ZXX coefficients (repeated calls to FlxX_to_ZXX).

GEN FlxXM_to_ZXXM(GEN B), converts an FlxXM to a t_MAT with ZXX coefficients (repeated calls to FlxX_to_ZXX).

GEN FlxC_to_ZXC(GEN x), converts a vector of Flx to a column vector of polynomials with t_INT coefficients (repeated calls to
 Flx_to_ZX).

GEN FlxV_to_ZXV(GEN x), as above but return a t_VEC.

void F2xV_to_FlxV_inplace(GEN v) v is destroyed.

void F2xV_to_ZXV_inplace(GEN v) v is destroyed.

void FlxV_to_ZXV_inplace(GEN v) v is destroyed.

GEN FlxM_to_ZXM(GEN z), converts a matrix of Flx to a matrix of polynomials with t_INT coefficients (repeated calls to Flx_to_ZX).

GEN zx_to_ZX(GEN z), as Flx_to_ZX, without assuming the coefficients to be non-negative.

GEN zx_to_Flx(GEN z, ulong p) as Flx_red without assuming the coefficients to be non-negative.

GEN Flc_to_ZC(GEN z), converts to ZC (t_COL of non-negative t_INTs in this case)

GEN Flc_to_ZC_inplace(GEN z), same as Flc_to_ZC, in place (z is destroyed).

GEN Flv_to_ZV(GEN z), converts to ZV (t_VEC of non-negative t_INTs in this case)

GEN Flm_to_ZM(GEN z), converts to ZM (t_MAT with non-negative t_INTs coefficients in this case)

GEN Flm_to_ZM_inplace(GEN z), same as Flm_to_ZM, in place (z is destroyed).

GEN zc_to_ZC(GEN z) as Flc_to_ZC, without assuming coefficients are non-negative.

GEN zv_to_ZV(GEN z) as Flv_to_ZV, without assuming coefficients are non-negative.

GEN zm_to_ZM(GEN z) as Flm_to_ZM, without assuming coefficients are non-negative.

GEN zv_to_Flv(GEN z, ulong p)

GEN zm_to_Flm(GEN z, ulong p)

7.3.43.3 Mixed precision linear algebra. Assumes dimensions are compatible. Multiply a multiprecision object by a single-precision one.

GEN RgM_zc_mul(GEN x, GEN y)

GEN RgMrow_zc_mul(GEN x, GEN y, long i)

GEN RgM_zm_mul(GEN x, GEN y)

GEN RgV_zc_mul(GEN x, GEN y)

GEN RgV_zm_mul(GEN x, GEN y)

GEN ZM_zc_mul(GEN x, GEN y)

GEN zv_ZM_mul(GEN x, GEN y)

GEN ZV_zc_mul(GEN x, GEN y)

GEN ZM_zm_mul(GEN x, GEN y)

GEN ZC_z_mul(GEN x, long y)

GEN ZM_nm_mul(GEN x, GEN y) the entries of y are ulongs.

GEN nm_Z_mul(GEN y, GEN c) the entries of y are ulongs.

7.3.43.4 Miscellaneous involving Fl.

GEN Fl_to_Flx(ulong x, long evx) converts a unsigned long to a scalar Flx. Assume that $evx = \text{evalvarn}(vx)$ for some variable number vx .

GEN Z_to_Flx(GEN x, ulong p, long sv) converts a t_INT to a scalar Flx polynomial. Assume that $sv = \text{evalvarn}(v)$ for some variable number v .

GEN Flx_to_Flv(GEN x, long n) converts from Flx to Flv with n components (assumed larger than the number of coefficients of x).

GEN zx_to_zv(GEN x, long n) as Flx_to_Flv.

GEN Flv_to_Flx(GEN x, long sv) converts from vector (coefficient array) to (normalized) polynomial in variable v .

GEN zv_to_zx(GEN x, long n) as Flv_to_Flx.

GEN Flm_to_FlxV(GEN x, long sv) converts the columns of Flm x to an array of Flx in the variable v (repeated calls to Flv_to_Flx).

GEN zm_to_zxV(GEN x, long n) as Flm_to_FlxV.

GEN Flm_to_FlxX(GEN x, long sw, long sv) same as Flm_to_FlxV(x, sv) but returns the result as a (normalized) polynomial in variable w .

GEN FlxV_to_Flm(GEN v, long n) reverse Flm_to_FlxV, to obtain an Flm with n rows (repeated calls to Flx_to_Flv).

GEN FlxX_to_Flx(GEN P) Let $P(x, X)$ be a FlxX, return $P(0, X)$ as a Flx.

GEN FlxX_to_Flm(GEN v, long n) reverse Flm_to_FlxX, to obtain an Flm with n rows (repeated calls to Flx_to_Flv).

GEN FlxX_to_FlxC(GEN B, long n, long sv) see RgX_to_RgV. The coefficients of B are assumed to be in the variable v.

GEN FlxXV_to_FlxM(GEN V, long n, long sv) see RgXV_to_RgM. The coefficients of V[i] are assumed to be in the variable v.

GEN Fly_to_FlxY(GEN a, long sv) convert coefficients of a to constant Flx in variable v.

7.3.43.5 Miscellaneous involving F2x.

GEN F2x_to_F2v(GEN x, long n) converts from F2x to F2v with n components (assumed larger than the number of coefficients of x).

GEN F2xC_to_ZXC(GEN x), converts a vector of F2x to a column vector of polynomials with t_INT coefficients (repeated calls to F2x_to_ZX).

GEN F2xC_to_FlxC(GEN x)

GEN FlxC_to_F2xC(GEN x)

GEN F2xV_to_F2m(GEN v, long n) F2x_to_F2v to each polynomial to get an F2m with n rows.

7.4 Higher arithmetic over Z: primes, factorization.

7.4.1 Pure powers.

long Z_issquare(GEN n) returns 1 if the t_INT n is a square, and 0 otherwise. This is tested first modulo small prime powers, then sqrtremi is called.

long Z_issquareall(GEN n, GEN *sqrtn) as Z_issquare. If n is indeed a square, set sqrtn to its integer square root. Uses a fast congruence test mod $64 \times 63 \times 65 \times 11$ before computing an integer square root.

long Z_ispow2(GEN x) returns 1 if the t_INT x is a power of 2, and 0 otherwise.

long uissquare(ulong n) as Z_issquare, for an ulong operand n.

long uissquareall(ulong n, ulong *sqrtn) as Z_issquareall, for an ulong operand n.

ulong usqrt(ulong a) returns the floor of the square root of a.

ulong usqrtn(ulong a, ulong n) returns the floor of the n-th root of a.

long Z_ispower(GEN x, ulong k) returns 1 if the t_INT n is a k-th power, and 0 otherwise; assume that $k > 1$.

long Z_ispowerall(GEN x, ulong k, GEN *pt) as Z_ispower. If n is indeed a k-th power, set *pt to its integer k-th root.

long Z_isanypower(GEN x, GEN *ptn) returns the maximal $k \geq 2$ such that the t_INT $x = n^k$ is a perfect power, or 0 if no such k exist; in particular ispower(1), ispower(0), ispower(-1) all return 0. If the return value k is not 0 (so that $x = n^k$) and ptn is not NULL, set *ptn to n.

The following low-level functions are called by Z_isanypower but can be directly useful:

`int is_357_power(GEN x, GEN *ptn, ulong *pmask)` tests whether the integer $x > 0$ is a 3-rd, 5-th or 7-th power. The bits of `*pmask` initially indicate which test is to be performed; bit 0: 3-rd, bit 1: 5-th, bit 2: 7-th (e.g. `*pmask = 7` performs all tests). They are updated during the call: if the “ i -th power” bit is set to 0 then x is not a k -th power. The function returns 0 (not a 3-rd, 5-th or 7-th power), 3 (3-rd power, not a 5-th or 7-th power), 5 (5-th power, not a 7-th power), or 7 (7-th power); if an i -th power bit is initially set to 0, we take it at face value and assume x is not an i -th power without performing any test. If the return value k is non-zero, set `*ptn` to n such that $x = n^k$.

`int is_pth_power(GEN x, GEN *ptn, forprime_t *T, ulong cutoff)` let $x > 0$ be an integer, `cutoff` > 0 and T be an iterator over primes ≥ 11 , we look for the smallest prime p such that $x = n^p$ (advancing T as we go along). The 11 is due to the fact that `is_357_power` and `issquare` are faster than the generic version for $p < 11$.

Fail and return 0 when the existence of p would imply $2^{\text{cutoff}} > x^{1/p}$, meaning that a possible n is so small that it should have been found by trial division; for maximal speed, you should start by a round of trial division, but the cut-off may also be set to 1 for a rigorous result without any trial division.

Otherwise returns the smallest suitable prime power p^i and set `*ptn` to the p^i -th root of x (which is now not a p -th power). We may immediately recall the function with the same parameters after setting $x = \text{*ptn}$: it will start at the next prime.

7.4.2 Factorization.

`GEN Z_factor(GEN n)` factors the `t_INT` n . The “primes” in the factorization are actually strong pseudoprimes.

`GEN absZ_factor(GEN n)` returns `Z_factor(absi(n))`.

`long Z_issmooth(GEN n, ulong lim)` returns 1 if all the prime factors of the `t_INT` n are less or equal to lim .

`GEN Z_issmooth_fact(GEN n, ulong lim)` returns NULL if a prime factor of the `t_INT` n is $> lim$, and returns the factorization of n otherwise, as a `t_MAT` with `t_VECSMALL` columns (word-size primes and exponents). Neither memory-clean nor suitable for `gerepileupto`.

`GEN Z_factor_until(GEN n, GEN lim)` as `Z_factor`, but stop the factorization process as soon as the unfactored part is smaller than lim . The resulting factorization matrix only contains the factors found. No other assumptions can be made on the remaining factors.

`GEN Z_factor_limit(GEN n, ulong lim)` trial divide n by all primes $p < lim$ in the precomputed list of prime numbers and return the corresponding factorization matrix. In this case, the last “prime” divisor in the first column of the factorization matrix may well be a proven composite.

If $lim = 0$, the effect is the same as setting $lim = \text{maxprime}() + 1$: use all precomputed primes.

`GEN absZ_factor_limit(GEN n, ulong all)` returns `Z_factor_limit(absi(n))`.

`GEN boundfact(GEN x, ulong lim)` as `Z_factor_limit`, applying to `t_INT` or `t_FRAC` inputs.

`GEN Z_smoother(GEN n, GEN L, GEN *pP, GEN *pE)` given a `t_VECSMALL` L containing a list of small primes and a `t_INT` n , trial divide n by the elements of L and return the cofactor. Return NULL if the cofactor is ± 1 . `*P` and `*E` contain the list of prime divisors found and their exponents, as `t_VECSMALLs`. Neither memory-clean, nor suitable for `gerepileupto`.

GEN `Z_factor_listP`(GEN `N`, GEN `L`) given a `t_INT` `N`, a vector or primes `L` containing all prime divisors of `N` (and possibly others). Return `factor(N)`. Neither memory-clean, nor suitable for `gerepileupto`.

GEN `factor_pn_1`(GEN `p`, `ulong n`) returns the factorization of $p^n - 1$, where p is prime and n is a positive integer.

GEN `factor_pn_1_limit`(GEN `p`, `ulong n`, `ulong B`) returns a partial factorization of $p^n - 1$, where p is prime and n is a positive integer. Don't actively search for prime divisors $p > B$, but we may still find some due to Aurifeuillian factorizations. Any entry $> B^2$ in the output factorization matrix is *a priori* not a prime (but may well be).

GEN `factor_Aurifeuille_prime`(GEN `p`, `long n`) an Aurifeuillian factor of $\phi_n(p)$, assuming p prime and an Aurifeuillian factor exists ($p\zeta_n$ is a square in $\mathbf{Q}(\zeta_n)$).

GEN `factor_Aurifeuille`(GEN `a`, `long d`) an Aurifeuillian factor of $\phi_n(a)$, assuming a is a non-zero integer and $n > 2$. Returns 1 if no Aurifeuillian factor exists.

GEN `odd_prime_divisors`(GEN `a`) `t_VEC` of all prime divisors of the `t_INT` `a`.

GEN `factoru`(`ulong n`), returns the factorization of n . The result is a 2-component vector $[P, E]$, where P and E are `t_VECSMALL` containing the prime divisors of n , and the $v_p(n)$.

GEN `factoru_pow`(`ulong n`), returns the factorization of n . The result is a 3-component vector $[P, E, C]$, where P, E and C are `t_VECSMALL` containing the prime divisors of n , the $v_p(n)$ and the $p^{v_p(n)}$.

GEN `vecfactoru`(`ulong a`, `ulong b`), returns a `t_VEC` v containing the factorizations (`factoru` format) of a, \dots, b ; assume that $b \geq a > 0$. Uses a sieve with primes up to \sqrt{b} . For all $c, a \leq c \leq b$, the factorization of c is given in $v[c - a + 1]$.

GEN `vecfactoroddu`(`ulong a`, `ulong b`), returns a `t_VEC` v containing the factorizations (`factoru` format) of odd integers in a, \dots, b ; assume that $b \geq a > 0$ are odd. Uses a sieve with primes up to \sqrt{b} . For all odd $c, a \leq c \leq b$, the factorization of c is given in $v[(c - a)/2 + 1]$.

GEN `vecfactoru_i`(`ulong a`, `ulong b`), private version of `vecfactoru`, not memory clean.

GEN `vecfactoroddu_i`(`ulong a`, `ulong b`), private version of `vecfactoroddu`, not memory clean.

GEN `vecfactorsquarefreeu`(`ulong a`, `ulong b`) return a `t_VEC` v containing the prime divisors of squarefree integers in a, \dots, b ; assume that $a \leq b$. Uses a sieve with primes up to \sqrt{b} . For all squarefree $c, a \leq c \leq b$, the prime divisors of c (as a `t_VECSMALL`) are given in $v[c - a + 1]$, and the other entries are `NULL`. Note that because of these `NULL` markers, v is not a valid `GEN`, it is not memory clean and cannot be used in garbage collection routines.

GEN `vecsquarefreeu`(`ulong a`, `ulong b`) return a `t_VECSMALL` v containing the squarefree integers in a, \dots, b . Assume that $a \leq b$. Uses a sieve with primes up to \sqrt{b} .

`ulong tridiv_bound`(GEN `n`) returns the trial division bound used by `Z_factor(n)`.

GEN `Z_pollardbrent`(GEN `N`, `long n`, `long seed`) try to factor `t_INT` `N` using $n \geq 1$ rounds of Pollard iterations; `seed` is an integer whose value (mod 8) selects the quadratic polynomial use to generate Pollard's (pseudo)random walk. Returns `NULL` on failure, else a vector of 2 (possibly 3) integers whose product is `N`.

GEN `Z_ECM`(GEN `N`, `long n`, `long seed`, `ulong B1`) try to factor `t_INT` `N` using $n \geq 1$ rounds of ECM iterations (on 8 to 64 curves simultaneously, depending on the size of `N`); `seed` is an integer

whose value selects the curves to be used: increase it by $64n$ to make sure that a subsequent call with a factor of N uses a disjoint set of curves. Finally $B_1 > 7$ determines the computations performed on the curves: we compute $[k]P$ for some point in $E(\mathbf{Z}/N\mathbf{Z})$ and $k = q \prod p^{e_p}$ where $p^{e_p} \leq B_1$ and $q \leq B_2 := 110B_1$; a higher value of B_1 means higher chances of hitting a factor and more time spent. The computation is deterministic for a given set of parameters. Returns NULL on failure, else a non trivial factor or N.

GEN `Q_factor`(GEN `x`) as `Z_factor`, where `x` is a `t_INT` or a `t_FRAC`.

GEN `Q_factor_limit`(GEN `x`, `ulong lim`) as `Z_factor_limit`, where `x` is a `t_INT` or a `t_FRAC`.

7.4.3 Coprime factorization.

Given a and b two non-zero integers, let `ppi`(a, b), `ppo`(a, b), `ppg`(a, b), `pple`(a, b) (powers in a of primes inside b , outside b , greater than those in b , less than or equal to those in b) be the integers defined by

- $v_p(\text{ppi}) = v_p(a)[v_p(b) > 0]$,
- $v_p(\text{ppo}) = v_p(a)[v_p(b) = 0]$,
- $v_p(\text{ppg}) = v_p(a)[v_p(a) > v_p(b)]$,
- $v_p(\text{pple}) = v_p(a)[v_p(a) \leq v_p(b)]$.

GEN `Z_ppo`(GEN `a`, GEN `b`) returns `ppo(a, b)`; shallow function.

`ulong u_ppo`(`ulong a`, `ulong b`) returns `ppo(a, b)`.

GEN `Z_ppgle`(GEN `a`, GEN `b`) returns [`ppg(a, b)`, `pple(a, b)`]; shallow function.

GEN `Z_ppio`(GEN `a`, GEN `b`) returns [`gcd(a, b)`, `ppi(a, b)`, `ppo(a, b)`]; shallow function.

GEN `Z_cba`(GEN `a`, GEN `b`) fast natural coprime base algorithm. Returns a vector of coprime divisors of a and b such that both a and b can be multiplicatively generated from this set. Perfect powers are not removed, is `Z_isanypower` if needed; shallow function.

GEN `ZV_cba_extend`(GEN `P`, GEN `b`) extend a coprime basis P by the integer b , the result being a coprime basis for $P \cup \{b\}$. Perfect powers are not removed; shallow function.

GEN `ZV_cba`(GEN `v`) given a vector of non-zero integers v , return a coprime basis for v . Perfect powers are not removed; shallow function.

7.4.4 Checks attached to arithmetic functions.

Arithmetic functions accept arguments of the following kind: a plain positive integer N (`t_INT`), the factorization fa of a positive integer (a `t_MAT` with two columns containing respectively primes and exponents), or a vector $[N, fa]$. A few functions accept non-zero integers (e.g. `omega`), and some others arbitrary integers (e.g. `factorint`, ...).

`int is_Z_factorpos`(GEN `f`) returns 1 if f looks like the factorization of a positive integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are positive integers. It does *not* check that entries in the first column (“primes”) are prime, or even pairwise coprime, nor that they are strictly increasing.

`int is_Z_factornon0`(GEN `f`) returns 1 if f looks like the factorization of a non-zero integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof, analogous to `is_Z_factorpos`. (Entries in the first column need only be non-zero integers.)

`int is_Z_factor(GEN f)` returns 1 if f looks like the factorization of an integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that f is a two-column matrix all of whose entries are integers. Entries in the second column (“exponents”) are all positive. Either it encodes the “factorization” 0^e , $e > 0$, or entries in the first column (“primes”) are all non-zero.

`GEN clean_Z_factor(GEN f)` assuming f is the factorization of an integer n , return the factorization of $|n|$, i.e. remove -1 from the factorization. Shallow function.

`GEN fuse_Z_factor(GEN f, GEN B)` assuming f is the factorization of an integer n , return `boundfact(n, B)`, i.e. return a factorization where all primary factors for $|p| \leq B$ are preserved, and all others are “fused” into a single composite integer; if that remainder is trivial, i.e. equal to 1, it is of course not included. Shallow function.

In the following three routines, f is the name of an arithmetic function, and n a supplied argument. They all raise exceptions if n does not correspond to an integer or an integer factorization of the expected shape.

`GEN check_arith_pos(GEN n, const char *f)` check whether n is attached to the factorization of a positive integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_DOMAIN` ($n \leq 0$) or an `e_TYPE` exception (other failures).

`GEN check_arith_non0(GEN n, const char *f)` check whether n is attached to the factorization of a non-0 integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise. May raise an `e_TYPE` exception.

`GEN check_arith_all(GEN n, const char *f)` is attached to the factorization of an integer, and return `NULL` (plain `t_INT`) or a factorization extracted from n otherwise.

7.4.5 Incremental integer factorization.

Routines attached to the dynamic factorization of an integer n , iterating over successive prime divisors. This is useful to implement high-level routines allowed to take shortcuts given enough partial information: e.g. `moebius(n)` can be trivially computed if we hit p such that $p^2 \mid n$. For efficiency, trial division by small primes should have already taken place. In any case, the functions below assume that no prime $< 2^{14}$ divides n .

`GEN ifac_start(GEN n, int moebius)` schedules a new factorization attempt for the integer n . If `moebius` is non-zero, the factorization will be aborted as soon as a repeated factor is detected (Moebius mode). The function assumes that $n > 1$ is a *composite* `t_INT` whose prime divisors satisfy $p > 2^{14}$ and that one can write to n in place.

This function stores data on the stack, no `gerepile` call should delete this data until the factorization is complete. Returns `partial`, a data structure recording the partial factorization state.

`int ifac_next(GEN *partial, GEN *p, long *e)` deletes a primary factor p^e from `partial` and sets `p` (prime) and `e` (exponent), and normally returns 1. Whatever remains in the `partial` structure is now coprime to p .

Returns 0 if all primary factors have been used already, so we are done with the factorization. In this case p is set to `NULL`. If we ran in Moebius mode and the factorization was in fact aborted, we have $e = 1$, otherwise $e = 0$.

`int ifac_read(GEN part, GEN *k, long *e)` peeks at the next integer to be factored in the list k^e , where k is not necessarily prime and can be a perfect power as well, but will be factored by the next call to `ifac_next`. You can remove this factorization from the schedule by calling:

`void ifac_skip(GEN part)` removes the next scheduled factorization.

`int ifac_isprime(GEN n)` given n whose prime divisors are $> 2^{14}$, returns the decision the factoring engine would take about the compositeness of n : 0 if n is a proven composite, and 1 if we believe it to be prime; more precisely, n is a proven prime if `factor_proven` is set, and only a BPSW-pseudoprime otherwise.

7.4.6 Integer core, squarefree factorization.

`long Z_issquarefree(GEN n)` returns 1 if the `t_INT` n is square-free, and 0 otherwise.

`long Z_isfundamental(GEN x)` returns 1 if the `t_INT` x is a fundamental discriminant, and 0 otherwise.

`GEN core(GEN n)` unique squarefree integer d dividing n such that n/d is a square. The core of 0 is defined to be 0.

`GEN core2(GEN n)` return $[d, f]$ with d squarefree and $n = df^2$.

`GEN corepartial(GEN n, long lim)` as `core`, using `boundfact(n, lim)` to partially factor n . The result is not necessarily squarefree, but $p^2 \mid n$ implies $p > \text{lim}$.

`GEN core2partial(GEN n, long lim)` as `core2`, using `boundfact(n, lim)` to partially factor n . The resulting d is not necessarily squarefree, but $p^2 \mid n$ implies $p > \text{lim}$.

7.4.7 Primes, primality and compositeness tests.

7.4.7.1 Chebyshev's π function, bounds.

`ulong uprimepi(ulong n)`, returns the number of primes $p \leq n$ (Chebyshev's π function).

`double primepi_upper_bound(double x)` return a quick upper bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_upper_bound(GEN x)` as `primepi_upper_bound`, returns a `t_REAL`.

`double primepi_lower_bound(double x)` return a quick lower bound for $\pi(x)$, using Dusart bounds.

`GEN gprimepi_lower_bound(GEN x)` as `primepi_lower_bound`, returns a `t_REAL` or `gen_0`.

7.4.7.2 Primes, primes in intervals.

`ulong unextprime(ulong n)`, returns the smallest prime $\geq n$. Return 0 if it cannot be represented as an `ulong` (n bigger than $2^{64} - 59$ or $2^{32} - 5$ depending on the word size).

`ulong uprecprime(ulong n)`, returns the largest prime $\leq n$. Return 0 if $n \leq 1$.

`ulong uprime(long n)` returns the n -th prime, assuming it fits in an `ulong` (overflow error otherwise).

`GEN prime(long n)` same as `utoi(uprime(n))`.

`GEN primes_zv(long m)` returns the first m primes, in a `t_VECSMALL`.

`GEN primes(long m)` return the first m primes, as a `t_VEC` of `t_INTs`.

GEN `primes_interval`(GEN `a`, GEN `b`) return the primes in the interval $[a, b]$, as a `t_VEC` of `t_INT`s.

GEN `primes_interval_zv`(ulong `a`, ulong `b`) return the primes in the interval $[a, b]$, as a `t_VECSMALL` of `ulongss`.

GEN `primes_upto_zv`(ulong `b`) return the primes in the interval $[2, b]$, as a `t_VECSMALL` of `ulongss`.

7.4.7.3 Tests.

int `uisprime`(ulong `p`), returns 1 if `p` is a prime number and 0 otherwise.

int `uisprime_101`(ulong `p`), assuming that p has no divisor ≤ 101 , returns 1 if `p` is a prime number and 0 otherwise.

int `uisprime_661`(ulong `p`), assuming that p has no divisor ≤ 661 , returns 1 if `p` is a prime number and 0 otherwise.

int `isprime`(GEN `n`), returns 1 if the `t_INT` `n` is a (fully proven) prime number and 0 otherwise.

long `isprimeAPRCL`(GEN `n`), returns 1 if the `t_INT` `n` is a prime number and 0 otherwise, using only the APRCL test — not even trial division or compositeness tests. The workhorse `isprime` should be faster on average, especially if non-primes are included!

long `isprimeECP`(GEN `n`), returns 1 if the `t_INT` `n` is a prime number and 0 otherwise, using only the ECP test. The workhorse `isprime` should be faster on average.

long `BPSW_psp`(GEN `n`), returns 1 if the `t_INT` `n` is a Baillie-Pomerance-Selfridge-Wagstaff pseudoprime, and 0 otherwise (proven composite).

int `BPSW_isprime`(GEN `x`) assuming x is a BPSW-pseudoprime, rigorously prove its primality. The function `isprime` is currently implemented as

```
BPSW_psp(x) && BPSW_isprime(x)
```

long `millerrabin`(GEN `n`, long `k`) performs k strong Rabin-Miller compositeness tests on the `t_INT` `n`, using k random bases. This function also caches square roots of -1 that are encountered during the successive tests and stops as soon as three distinct square roots have been produced; we have in principle factored n at this point, but unfortunately, there is currently no way for the factoring machinery to become aware of it. (It is highly implausible that hard to find factors would be exhibited in this way, though.) This should be slower than `BPSW_psp` for $k \geq 4$ and we would expect it to be less reliable.

GEN `ecpp`(GEN `N`) returns an ECPP certificate for `t_INT` `N`; underlies `primecert`.

GEN `ecppexport`(GEN `cert`, long `flag`) export a PARI ECPP certificate to MAGMA or Primo format; underlies `primecertexport`.

long `ecppisvalid`(GEN `cert`) checks whether a PARI ECPP certificate is valid; underlies `primecertisvalid`.

7.4.8 Iterators over primes.

`int forprime_init(forprime_t *T, GEN a, GEN b)` initialize an iterator T over primes in $[a, b]$; over primes $\geq a$ if $b = \text{NULL}$. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator T over primes in an arithmetic progression in $[a, b]$; over primes $\geq a$ if $b = \text{NULL}$. The argument q is either a `t_INT` ($p \equiv a \pmod{q}$) or a `t_INTMOD` $\text{Mod}(c, N)$ and we restrict to that congruence class. Return 0 if the range is known to be empty from the start (as if $b < a$ or $b < 0$), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`GEN forprime_next(forprime_t *T)` returns the next prime in the range, assuming that T was initialized by `forprime_init`.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)`

`ulong u_forprime_next(forprime_t *T)`

`void u_forprime_restrict(forprime_t *T, ulong c)` let T an iterator over primes initialized via `u_forprime_init(&T, a, b)`, possibly followed by a number of calls to `u_forprime_next`, and $a \leq c \leq b$. Restrict the range of primes considered to $[a, c]$.

`int u_forprime_arith_init(forprime_t *T, ulong a, ulong b, ulong c, ulong q)` initialize an iterator over primes in $[a, b]$, congruent to c modulo q . Subsequent calls to `u_forprime_next` will only return primes congruent to c modulo q . Note that unless $(c, q) = 1$ there will be at most one such prime.

7.5 Integral, rational and generic linear algebra.

7.5.1 ZC / ZV, ZM. A ZV (resp. a ZM, resp. a ZX) is a `t_VEC` or `t_COL` (resp. `t_MAT`, resp. `t_POL`) with `t_INT` coefficients.

7.5.1.1 ZC / ZV.

`void RgV_check_ZV(GEN x, const char *s)` Assuming x is a `t_VEC` or `t_COL` raise an error if it is not a ZV (s should point to the name of the caller).

`int RgV_is_ZV(GEN x)` Assuming x is a `t_VEC` or `t_COL` return 1 if it is a ZV, and 0 otherwise.

`int RgV_is_ZVpos(GEN x)` Assuming x is a `t_VEC` or `t_COL` return 1 if it is a ZV with positive entries, and 0 otherwise.

`int RgV_is_ZVnon0(GEN x)` Assuming x is a `t_VEC` or `t_COL` return 1 if it is a ZV with non-zero entries, and 0 otherwise.

`int RgV_is_QV(GEN P)` return 1 if the `RgV P` has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int ZV_equal0(GEN x)` returns 1 if all entries of the ZV x are zero, and 0 otherwise.

`int ZV_cmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order, comparing absolute values).

`int ZV_abcmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order).

`int ZV_equal(GEN x, GEN y)` returns 1 if the two ZV are equal and 0 otherwise. A `t_COL` and a `t_VEC` with the same entries are declared equal.

`GEN ZC_add(GEN x, GEN y)` adds x and y .

`GEN ZC_sub(GEN x, GEN y)` subtracts x and y .

`GEN ZC_Z_add(GEN x, GEN y)` adds y to $x[1]$.

`GEN ZC_Z_sub(GEN x, GEN y)` subtracts y to $x[1]$.

`GEN Z_ZC_sub(GEN a, GEN x)` returns the vector $[a - x_1, -x_2, \dots, -x_n]$.

`GEN ZC_copy(GEN x)` returns a (`t_COL`) copy of x .

`GEN ZC_neg(GEN x)` returns $-x$ as a `t_COL`.

`void ZV_neg_inplace(GEN x)` negates the ZV x in place, by replacing each component by its opposite (the type of x remains the same, `t_COL` or `t_ROW`). If you want to save even more memory by avoiding the implicit component copies, use `ZV_togglesign`.

`void ZV_togglesign(GEN x)` negates x in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZC_Z_mul(GEN x, GEN y)` multiplies the ZC or ZV x (which can be a column or row vector) by the `t_INT` y , returning a ZC.

`GEN ZC_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZC_Z_div(GEN x, GEN y)` returns x/y , where the resulting vector has rational entries.

`GEN ZV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming x and y have `t_INT` entries.

`GEN ZV_dotsquare(GEN x)` as `RgV_dotsquare` assuming x has `t_INT` entries.

`GEN ZC_lincomb(GEN u, GEN v, GEN x, GEN y)` returns $ux + vy$, where u, v are `t_INT` and x, y are ZC or ZV. Return a ZC

`void ZC_lincomb1_inplace(GEN X, GEN Y, GEN v)` sets $X \leftarrow X + vY$, where v is a `t_INT` and X, Y are ZC or ZV. (The result has the type of X .) Memory efficient (e.g. no-op if $v = 0$), but not gerepile-safe.

`void ZC_lincomb1_inplace_i(GEN X, GEN Y, GEN v, long n)` variant of `ZC_lincomb1_inplace`: only update $X[1], \dots, X[n]$, assuming that $n < \lg(X)$.

`GEN ZC_ZV_mul(GEN x, GEN y, GEN p)` multiplies the ZC x (seen as a column vector) by the ZV y (seen as a row vector, assumed to have compatible dimensions).

`GEN ZV_content(GEN x)` returns the GCD of all the components of x .

`GEN ZV_extgcd(GEN A)` given a vector of n integers A , returns $[d, U]$, where d is the content of A and U is a matrix in $GL_n(\mathbf{Z})$ such that $AU = [D, 0, \dots, 0]$.

`GEN ZV_prod(GEN x)` returns the product of all the components of x (1 for the empty vector).

`GEN ZV_sum(GEN x)` returns the sum of all the components of x (0 for the empty vector).

`long ZV_max_lg(GEN x)` returns the effective length of the longest entry in x .

`int ZV_dvd(GEN x, GEN y)` assuming x, y are two ZVs of the same length, return 1 if $y[i]$ divides $x[i]$ for all i and 0 otherwise. Error if one of the $y[i]$ is 0.

`GEN ZV_sort(GEN L)` sort the ZV L . Returns a vector with the same type as L .

`void ZV_sort_inplace(GEN L)` sort the ZV L , in place.

`GEN ZV_sort_uniq(GEN L)` sort the ZV L , removing duplicate entries. Returns a vector with the same type as L .

`long ZV_search(GEN L, GEN y)` look for the `t_INT` y in the sorted ZV L . Return an index i such that $L[i] = y$, and 0 otherwise.

`GEN ZV_indexsort(GEN L)` returns the permutation which, applied to the ZV L , would sort the vector. The result is a `t_VECSMALL`.

`GEN ZV_union_shallow(GEN x, GEN y)` given two *sorted* ZV (as per `ZV_sort`, returns the union of x and y . Shallow function. In case two entries are equal in x and y , include the one from x .

`GEN ZC_union_shallow(GEN x, GEN y)` as `ZV_union_shallow` but return a `t_COL`.

7.5.1.2 ZM.

`void RgM_check_ZM(GEN A, const char *s)` Assuming x is a `t_MAT` raise an error if it is not a ZM (s should point to the name of the caller).

`GEN RgM_rescale_to_int(GEN x)` given a matrix x with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a ZM which is very close to Dx for some well-chosen integer D . More precisely, if the input is exact, D is the denominator of x ; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

`GEN ZM_copy(GEN x)` returns a copy of x .

`int ZM_equal(GEN A, GEN B)` returns 1 if the two ZM are equal and 0 otherwise.

`int ZM_equal0(GEN A)` returns 1 if the ZM A is identically equal to 0.

`GEN ZM_add(GEN x, GEN y)` returns $x + y$ (assumed to have compatible dimensions).

`GEN ZM_sub(GEN x, GEN y)` returns $x - y$ (assumed to have compatible dimensions).

`GEN ZM_neg(GEN x)` returns $-x$.

`void ZM_togglesign(GEN x)` negates x in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZM_mul(GEN x, GEN y)` multiplies x and y (assumed to have compatible dimensions).

`GEN ZM_sqr(GEN x)` returns x^2 , where x is a square ZM.

`GEN ZM_Z_mul(GEN x, GEN y)` multiplies the ZM x by the `t_INT` y .

`GEN ZM_ZC_mul(GEN x, GEN y)` multiplies the ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions).

`GEN ZM_ZX_mul(GEN x, GEN T)` returns $x \times y$, where y is `RgX_to_RgC(T, lg(x) - 1)`.

`GEN ZM_diag_mul(GEN d, GEN m)` given a vector d with integer entries and a ZM m of compatible dimensions, return `diagonal(d) * m`.

`GEN ZM_mul_diag(GEN m, GEN d)` given a vector d with integer entries and a ZM m of compatible dimensions, return $m * \text{diagonal}(d)$.

`GEN ZM_multosym(GEN x, GEN y)`

`GEN ZM_transmultosym(GEN x, GEN y)`

`GEN ZM_transmul(GEN x, GEN y)`

`GEN ZMrow_ZC_mul(GEN x, GEN y, long i)` multiplies the i -th row of ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions). Assumes that x is non-empty and $0 < i < \text{lg}(x[1])$.

`GEN ZV_ZM_mul(GEN x, GEN y)` multiplies the ZV x by the ZM y . Returns a `t_VEC`.

`GEN ZM_Z_divexact(GEN x, GEN y)` returns x/y assuming all divisions are exact.

`GEN ZM_Z_div(GEN x, GEN y)` returns x/y , where the resulting matrix has rational entries.

`GEN ZC_Q_mul(GEN x, GEN y)` returns $x*y$, where y is a rational number and the resulting `t_COL` has rational entries.

`GEN ZM_Q_mul(GEN x, GEN y)` returns $x*y$, where y is a rational number and the resulting matrix has rational entries.

`GEN ZM_pow(GEN x, GEN n)` returns x^n , assuming x is a square ZM and $n \geq 0$.

`GEN ZM_powu(GEN x, ulong n)` returns x^n , assuming x is a square ZM and $n \geq 0$.

`GEN ZM_det(GEN M)` if M is a ZM, returns the determinant of M . This is the function underlying `matdet` whenever M is a ZM.

`GEN ZM_permanent(GEN M)` if M is a ZM, returns its permanent. This is the function underlying `matpermanent` whenever M is a ZM. It assumes that the matrix is square of dimension $< \text{BITS_IN_LONG}$.

`GEN ZM_detmult(GEN M)` if M is a ZM, returns a multiple of the determinant of the lattice generated by its columns. This is the function underlying `detint`.

`GEN ZM_supnorm(GEN x)` return the sup norm of the ZM x .

`GEN ZM_charpoly(GEN M)` returns the characteristic polynomial (in variable 0) of the ZM M .

`GEN ZM_imagecompl(GEN x)` returns `matimagecompl(x)`.

`long ZM_rank(GEN x)` returns `matrank(x)`.

`GEN ZM_ker(GEN x)` returns `matker(x)`

`GEN ZM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN ZM_indeximage(GEN x)` returns `gel(ZM_indexrank(x), 2)`.

`long ZM_max_lg(GEN x)` returns the effective length of the longest entry in x .

`GEN ZM_inv(GEN M, GEN *pd)` if M is a ZM, return a primitive matrix H such that MH is d times the identity and set `*pd` to d . Uses a multimodular algorithm up to Hadamard's bound. If you suspect that the denominator is much smaller than $\det M$, you may use `ZM_inv_ratlift`.

`GEN ZM_inv_ratlift(GEN M, GEN *pd)` if M is a ZM, return a primitive matrix H such that MH is d times the identity and set `*pd` to d . Uses a multimodular algorithm, attempting rational

reconstruction along the way. To be used when you expect that the denominator of M^{-1} is much smaller than $\det M$ else use `ZM_inv`.

`GEN ZM_pseudoinv(GEN M, GEN *pv, GEN *pd)` if M is a non-empty ZM, let $v = [y, z]$ returned by `indexrank` and let M_1 be the corresponding square invertible matrix. Return a primitive left-inverse H such that HM_1 is d times the identity and set `*pd` to d . If `pv` is not NULL, set `*pv` to v . Not gerepile-safe.

`GEN ZM_gauss(GEN a, GEN b)` as `gauss`, where a and b coefficients are `t_INTs`.

`GEN ZM_det_triangular(GEN x)` returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

`int ZM_isidentity(GEN x)` return 1 if the ZM x is the identity matrix, and 0 otherwise.

`int ZM_isdiagonal(GEN x)` return 1 if the ZM x is diagonal, and 0 otherwise.

`int ZM_isscalar(GEN x, GEN s)` given a ZM x and a `t_INT` s , return 1 if x is equal to s times the identity, and 0 otherwise. If s is NULL, test whether x is an arbitrary scalar matrix.

`long ZC_is_ei(GEN x)` return i if the ZC x has 0 entries, but for a 1 at position i .

`int ZM_ishnf(GEN x)` return 1 if x is in HNF form, i.e. is upper triangular with positive diagonal coefficients, and for $j > i$, $x_{i,i} > x_{i,j} \geq 0$.

7.5.2 QM.

`GEN QM_charpoly_ZX(GEN M)` returns the characteristic polynomial (in variable 0) of the QM M , assuming that the result has integer coefficients.

`GEN QM_charpoly_ZX_bound(GEN M, long b)` as `QM_charpoly_ZX` assuming that the sup norm of the (integral) result is $\leq 2^b$.

`GEN QM_gauss(GEN a, GEN b)` as `gauss`, where a and b coefficients are `t_FRACs`.

`GEN QM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN QM_inv(GEN M)` return the inverse of the QM M .

`long QM_rank(GEN x)` returns `matrank(x)`.

7.5.3 Qevproj.

`GEN Qevproj_init(GEN M)` let M be a $n \times d$ ZM of maximal rank $d \leq n$, representing the basis of a \mathbf{Q} -subspace V of \mathbf{Q}^n . Return a projector on V , to be used by `Qevproj_apply`. The interface details may change in the future, but this function currently returns $[M, B, D, p]$, where p is a `t_VECSMALL` with d entries such that the submatrix $A = \text{rowpermute}(M, p)$ is invertible, B is a ZM and d a `t_INT` such that $AB = D\text{Id}_d$.

`GEN Qevproj_apply(GEN T, GEN pro)` let T be an $n \times n$ QM, stabilizing a \mathbf{Q} -subspace $V \subset \mathbf{Q}^n$ of dimension d , and let `pro` be a projector on that subspace initialized by `Qevproj_init(M)`. Return the $d \times d$ matrix representing $T|_V$ on the basis given by the columns of M .

`GEN Qevproj_apply_vecei(GEN T, GEN pro, long k)` as `Qevproj_apply`, return only the image of the k -th basis vector $M[k]$ (still on the basis given by the columns of M).

`GEN Qevproj_down(GEN T, GEN pro)` given a ZC (resp. a ZM) T representing an element (resp. a vector of elements) in the subspace V return a QC (resp. a QM) U such that $T = MU$.

7.5.4 zv, zm.

GEN `zv_neg(GEN x)` return $-x$. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\text{BITS_IN_LONG}-1}$.

GEN `zv_neg_inplace(GEN x)` negates x in place and return it. No check for overflow is done, which occurs in the fringe case where an entry is equal to $2^{\text{BITS_IN_LONG}-1}$.

GEN `zm_zc_mul(GEN x, GEN y)`

GEN `zm_mul(GEN x, GEN y)`

GEN `zv_z_mul(GEN x, long n)` return nx . No check for overflow is done.

long `zv_content(GEN x)` returns the gcd of the entries of x .

long `zv_dotproduct(GEN x, GEN y)`

long `zv_prod(GEN x)` returns the product of all the components of x (assumes no overflow occurs).

GEN `zv_prod_Z(GEN x)` returns the product of all the components of x ; consider all $x[i]$ as `ulongs`.

long `zv_sum(GEN x)` returns the sum of all the components of x (assumes no overflow occurs).

int `zv_cmp0(GEN x)` returns 1 if all entries of the `zv x` are 0, and 0 otherwise.

int `zv_equal(GEN x, GEN y)` returns 1 if the two `zv` are equal and 0 otherwise.

int `zv_equal0(GEN x)` returns 1 if all entries are 0, and return 0 otherwise.

long `zv_search(GEN L, long y)` look for y in the sorted `zv L`. Return an index i such that $L[i] = y$, and 0 otherwise.

GEN `zv_copy(GEN x)` as `Flv_copy`.

GEN `zm_transpose(GEN x)` as `Flm_transpose`.

GEN `zm_copy(GEN x)` as `Flm_copy`.

GEN `zero_zm(long m, long n)` as `zero_Flm`.

GEN `zero_zv(long n)` as `zero_Flv`.

GEN `zm_row(GEN A, long x0)` as `Flm_row`.

GEN `zm_permanent(GEN M)` return the permanent of M . The function assumes that the matrix is square of dimension $< \text{BITS_IN_LONG}$.

int `zvV_equal(GEN x, GEN y)` returns 1 if the two `zvV` (vectors of `zv`) are equal and 0 otherwise.

7.5.5 ZMV / zmV (vectors of ZM/zm).

int `RgV_is_ZMV(GEN x)` Assuming x is a `t_VEC` or `t_COL` return 1 if its components are ZM, and 0 otherwise.

GEN `ZMV_to_zmV(GEN z)`

GEN `zmV_to_ZMV(GEN z)`

GEN `ZMV_to_FlmV(GEN z, ulong m)`

7.5.6 QC / QV, QM.

GEN QM_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN QM_QC_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN QM_det(GEN M) returns the determinant of M .

GEN QM_ker(GEN x) returns `matker(x)`.

7.5.7 RgC / RgV, RgM.

RgC and RgV routines assume the inputs are VEC or COL of the same dimension. RgM assume the inputs are MAT of compatible dimensions.

7.5.7.1 Matrix arithmetic.

void RgM_dimensions(GEN x, long *m, long *n) sets m , resp. n , to the number of rows, resp. columns of the `t_MAT` x .

GEN RgC_add(GEN x, GEN y) returns $x + y$ as a `t_COL`.

GEN RgC_neg(GEN x) returns $-x$ as a `t_COL`.

GEN RgC_sub(GEN x, GEN y) returns $x - y$ as a `t_COL`.

GEN RgV_add(GEN x, GEN y) returns $x + y$ as a `t_VEC`.

GEN RgV_neg(GEN x) returns $-x$ as a `t_VEC`.

GEN RgV_sub(GEN x, GEN y) returns $x - y$ as a `t_VEC`.

GEN RgM_add(GEN x, GEN y) return $x + y$.

GEN RgM_neg(GEN x) returns $-x$.

GEN RgM_sub(GEN x, GEN y) returns $x - y$.

GEN RgM_Rg_add(GEN x, GEN y) assuming x is a square matrix and y a scalar, returns the square matrix $x + y * \text{Id}$.

GEN RgM_Rg_add_shallow(GEN x, GEN y) as RgM_Rg_add with much fewer copies. Not suitable for `gerepileupto`.

GEN RgM_Rg_sub(GEN x, GEN y) assuming x is a square matrix and y a scalar, returns the square matrix $x - y * \text{Id}$.

GEN RgM_Rg_sub_shallow(GEN x, GEN y) as RgM_Rg_sub with much fewer copies. Not suitable for `gerepileupto`.

GEN RgC_Rg_add(GEN x, GEN y) assuming x is a non-empty column vector and y a scalar, returns the vector $[x_1 + y, x_2, \dots, x_n]$.

GEN RgC_Rg_sub(GEN x, GEN y) assuming x is a non-empty column vector and y a scalar, returns the vector $[x_1 - y, x_2, \dots, x_n]$.

GEN Rg_RgC_sub(GEN a, GEN x) assuming x is a non-empty column vector and a a scalar, returns the vector $[a - x_1, -x_2, \dots, -x_n]$.

GEN RgC_Rg_div(GEN x, GEN y)

GEN RgM_Rg_div(GEN x, GEN y) returns x/y (y treated as a scalar).

GEN RgC_Rg_mul(GEN x, GEN y)
 GEN RgV_Rg_mul(GEN x, GEN y)
 GEN RgM_Rg_mul(GEN x, GEN y) returns $x \times y$ (y treated as a scalar).
 GEN RgV_RgC_mul(GEN x, GEN y) returns $x \times y$.
 GEN RgV_RgM_mul(GEN x, GEN y) returns $x \times y$.
 GEN RgM_RgC_mul(GEN x, GEN y) returns $x \times y$.
 GEN RgM_RgX_mul(GEN x, GEN T) returns $x \times y$, where y is `RgX_to_RgC(T, lg(x) - 1)`.
 GEN RgM_mul(GEN x, GEN y) returns $x \times y$.
 GEN RgM_transmul(GEN x, GEN y) returns $x^{\sim} \times y$.
 GEN RgM_multosym(GEN x, GEN y) returns $x \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).
 GEN RgM_transmultosym(GEN x, GEN y) returns $x^{\sim} \times y$, assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).
 GEN RgMrow_RgC_mul(GEN x, GEN y, long i) multiplies the i -th row of `RgM` x by the `RgC` y (seen as a column vector, assumed to have compatible dimensions). Assumes that x is non-empty and $0 < i < \text{lg}(x[1])$.
 GEN RgM_mulreal(GEN x, GEN y) returns the real part of $x \times y$ (whose entries are `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`).
 GEN RgM_sqr(GEN x) returns x^2 .
 GEN RgC_RgV_mul(GEN x, GEN y) returns $x \times y$ (the matrix $(x_i y_j)$).

The following two functions are not well defined in general and only provided for convenience in specific cases:

GEN RgC_RgM_mul(GEN x, GEN y) returns $x \times y[1,]$ if y is a row matrix $1 \times n$, error otherwise.
 GEN RgM_RgV_mul(GEN x, GEN y) returns $x \times y[, 1]$ if y is a column matrix $n \times 1$, error otherwise.
 GEN RgM_powers(GEN x, long n) returns $[x^0, \dots, x^n]$ as a `t_VEC` of `RgMs`.
 GEN RgV_sum(GEN v) sum of the entries of v
 GEN RgV_prod(GEN v) product of the entries of v , using a divide and conquer strategy
 GEN RgV_sumpart(GEN v, long n) returns the sum $v[1] + \dots + v[n]$ (assumes that $\text{lg}(v) > n$).
 GEN RgV_sumpart2(GEN v, long m, long n) returns the sum $v[m] + \dots + v[n]$ (assumes that $\text{lg}(v) > n$ and $m > 0$). Returns `gen_0` when $m > n$.
 GEN RgM_sumcol(GEN v) returns a `t_COL`, sum of the columns of the `t_MAT` v .
 GEN RgV_dotproduct(GEN x, GEN y) returns the scalar product of x and y
 GEN RgV_dotsquare(GEN x) returns the scalar product of x with itself.
 GEN RgV_killo(GEN v) returns a shallow copy of v where entries matched by `gequal0` are replaced by `NULL`. The return value is not a valid `GEN` and must be handled specially. The idea is to pre-treat a vector of coefficients to speed up later linear combinations or scalar products.

GEN `gram_matrix`(GEN `v`) returns the Gram matrix ($v_i \cdot v_j$) attached to the entries of v (matrix, or vector of vectors).

GEN `RgV_polint`(GEN `X`, GEN `Y`, long `v`) X and Y being two vectors of the same length, returns the polynomial T in variable v such that $T(X[i]) = Y[i]$ for all i . The special case $X = \text{NULL}$ corresponds to $X = [1, 2, \dots, n]$, where n is the length of Y .

7.5.7.2 Special shapes.

The following routines check whether matrices or vectors have a special shape, using `gequal1` and `gequal0` to test components. (This makes a difference when components are inexact.)

int `RgV_isscalar`(GEN `x`) return 1 if all the entries of x are 0 (as per `gequal0`), except possibly the first one. The name comes from vectors expressing polynomials on the standard basis $1, T, \dots, T^{n-1}$, or on `nf.zk` (whose first element is 1).

int `QV_isscalar`(GEN `x`) as `RgV_isscalar`, assuming x is a QV (`t_INT` and `t_FRAC` entries only).

int `ZV_isscalar`(GEN `x`) as `RgV_isscalar`, assuming x is a ZV (`t_INT` entries only).

int `RgM_isscalar`(GEN `x`, GEN `s`) return 1 if x is the scalar matrix equal to s times the identity, and 0 otherwise. If s is `NULL`, test whether x is an arbitrary scalar matrix.

int `RgM_isidentity`(GEN `x`) return 1 if the `t_MAT` x is the identity matrix, and 0 otherwise.

int `RgM_isdiagonal`(GEN `x`) return 1 if the `t_MAT` x is a diagonal matrix, and 0 otherwise.

long `RgC_is_ei`(GEN `x`) return i if the `t_COL` x has 0 entries, but for a 1 at position i .

int `RgM_is_ZM`(GEN `x`) return 1 if the `t_MAT` x has only `t_INT` coefficients, and 0 otherwise.

int `RgM_is_QM`(GEN `x`) return 1 if the `t_MAT` x has only `t_INT` or `t_FRAC` coefficients, and 0 otherwise.

long `RgV_isin`(GEN `v`, GEN `x`) return the first index i such that $v[i] = x$ if it exists, and 0 otherwise. Naive search in linear time, does not assume that v is sorted.

GEN `RgM_diagonal`(GEN `m`) returns the diagonal of m as a `t_VEC`.

GEN `RgM_diagonal_shallow`(GEN `m`) shallow version of `RgM_diagonal`

7.5.7.3 Conversion to floating point entries.

GEN `RgC_gtofp`(GEN `x`, GEN `prec`) returns the `t_COL` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x .

GEN `RgV_gtofp`(GEN `x`, GEN `prec`) returns the `t_VEC` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x .

GEN `RgC_gtomp`(GEN `x`, long `prec`) returns the `t_COL` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x .

GEN `RgC_fpnorml2`(GEN `x`, long `prec`) returns (a stack-clean variant of)

`gnorml2(RgC_gtofp(x, prec))`

GEN `RgM_gtofp`(GEN `x`, GEN `prec`) returns the `t_MAT` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of x .

GEN `RgM_gtomp`(GEN `x`, long `prec`) returns the `t_MAT` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of x .

GEN RgM_fpnorml2(GEN x, long prec) returns (a stack-clean variant of)

```
gnorml2( RgM_gtofp(x, prec) )
```

7.5.7.4 Linear algebra, linear systems.

GEN RgM_inv(GEN a) returns a left inverse of a (which needs not be square), or NULL if this turns out to be impossible. The latter happens when the matrix does not have maximal rank (or when rounding errors make it appear so).

GEN RgM_inv_upper(GEN a) as RgM_inv, assuming that a is a non-empty invertible upper triangular matrix, hence a little faster.

GEN RgM_RgC_invmage(GEN A, GEN B) returns a $\mathfrak{t_COL}$ X such that $AX = B$ if one such exists, and NULL otherwise.

GEN RgM_invmage(GEN A, GEN B) returns a $\mathfrak{t_MAT}$ X such that $AX = B$ if one such exists, and NULL otherwise.

GEN RgM_Hadamard(GEN a) returns an upper bound for the absolute value of $\det(a)$. The bound is a $\mathfrak{t_INT}$.

GEN RgM_solve(GEN a, GEN b) returns $a^{-1}b$ where a is a square $\mathfrak{t_MAT}$ and b is a $\mathfrak{t_COL}$ or $\mathfrak{t_MAT}$. Returns NULL if a^{-1} cannot be computed, see RgM_inv.

If $b = \text{NULL}$, the matrix a need no longer be square, and we strive to return a left inverse for a (NULL if it does not exist).

GEN RgM_solve_realimag(GEN M, GEN b) M being a $\mathfrak{t_MAT}$ with $r_1 + r_2$ rows and $r_1 + 2r_2$ columns, y a $\mathfrak{t_COL}$ or $\mathfrak{t_MAT}$ such that the equation $Mx = y$ makes sense, returns x under the following simplifying assumptions: the first r_1 rows of M and y are real (the r_2 others are complex), and x is real. This is stabler and faster than calling RgM_solve(M, b) over \mathbf{C} . In most applications, M approximates the complex embeddings of an integer basis in a number field, and x is actually rational.

GEN split_realimag(GEN x, long r1, long r2) x is a $\mathfrak{t_COL}$ or $\mathfrak{t_MAT}$ with $r_1 + r_2$ rows, whose first r_1 rows have real entries (the r_2 others are complex). Return an object of the same type as x and $r_1 + 2r_2$ rows, such that the first $r_1 + r_2$ rows contain the real part of x , and the r_2 following ones contain the imaginary part of the last r_2 rows of x . Called by RgM_solve_realimag.

GEN RgM_det_triangular(GEN x) returns the product of the diagonal entries of x (its determinant if it is indeed triangular).

GEN Frobeniusform(GEN V, long n) given the vector V of elementary divisors for $M - x\text{Id}$, where M is an $n \times n$ square matrix. Returns the Frobenius form of M .

int RgM_QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec) QR-decomposition of a square invertible $\mathfrak{t_MAT}$ x with real coefficients. Sets $*pB$ to the vector of squared lengths of the $x[i]$, $*pL$ to the Gram-Schmidt coefficients and $*pQ$ to a vector of successive Householder transforms. If R denotes the transpose of L and Q is the result of applying $*pQ$ to the identity matrix, then $x = QR$ is the QR decomposition of x . Returns 0 if x is not invertible or we hit a precision problem, and 1 otherwise.

int QR_init(GEN x, GEN *pB, GEN *pQ, GEN *pL, long prec) as RgM_QR_init, assuming further that x has $\mathfrak{t_INT}$ or $\mathfrak{t_REAL}$ coefficients.

GEN `R_from_QR`(GEN `x`, long `prec`) assuming that x is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, return the upper triangular R from the QR decomposition of x . Not memory clean. If the matrix is not known to have `t_INT` or `t_REAL` coefficients, apply `RgM_gtomp` first.

GEN `gaussred_from_QR`(GEN `x`, long `prec`) assuming that x is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, returns `qfgaussred(x~* x)`; this is essentially the upper triangular R matrix from the QR decomposition of x , renormalized to accommodate `qfgaussred` conventions. Not memory clean.

GEN `RgM_gram_schmidt`(GEN `e`, GEN `*ptB`) naive (unstable) Gram-Schmidt orthogonalization of the basis (e_i) given by the columns of `t_MAT` e . Return the e_i^* (as columns of a `t_MAT`) and set `*ptB` to the vector of squared lengths $|e_i^*|^2$.

GEN `RgM_Babai`(GEN `M`, GEN `y`) given a `t_MAT` M of maximal rank n and a `t_COL` y of the same dimension, apply Babai's nearest plane algorithm to return an *integral* x such that $y - Mx$ has small L_2 norm. This yields an approximate solution to the closest vector problem: if M is LLL-reduced, then

$$\|y - Mx\|_2 \leq 2(2/\sqrt{3})^n \|y - MX\|_2$$

for all $X \in \mathbf{Z}^n$.

7.5.8 ZG.

Let G be a multiplicative group with neutral element 1_G whose multiplication is supported by `gmul` and where equality test is performed using `gidentical`, e.g. a matrix group. The following routines implement basic computations in the group algebra $\mathbf{Z}[G]$. All of them are shallow for efficiency reasons. A ZG is either

- a `t_INT` n , representing $n[1_G]$
- or a “factorization matrix” with two columns $[g, e]$: the first one contains group elements, sorted according to `cmp_universal`, and the second one contains integer “exponents”, representing $\sum e_i [g_i]$.

Note that `to_famat` and `to_famat_shallow`(g, e) allow to build the ZG $e[g]$ from $e \in \mathbf{Z}$ and $g \in G$.

GEN `ZG_normalize`(GEN `x`) given a `t_INT` x or a factorization matrix *without* assuming that the first column is properly sorted. Return a valid (sorted) ZG. Shallow function.

GEN `ZG_add`(GEN `x`, GEN `y`) return $x + y$; shallow function.

GEN `ZG_neg`(GEN `x`) return $-x$; shallow function.

GEN `ZG_sub`(GEN `x`, GEN `y`) return $x - y$; shallow function.

GEN `ZG_mul`(GEN `x`, GEN `y`) return xy ; shallow function.

GEN `ZG_G_mul`(GEN `x`, GEN `y`) given a ZG x and $y \in G$, return xy ; shallow function.

GEN `G_ZG_mul`(GEN `x`, GEN `y`) given a ZG y and $x \in G$, return xy ; shallow function.

GEN `ZG_Z_mul`(GEN `x`, GEN `n`) given a ZG x and $y \in \mathbf{Z}$, return xy ; shallow function.

GEN `ZGC_G_mul`(GEN `v`, GEN `x`) given v a vector of ZG and $x \in G$ return the vector (with the same type as v with entries $v[i] \cdot x$). Shallow function.

void `ZGC_G_mul_inplace`(GEN `v`, GEN `x`) as `ZGC_G_mul`, modifying v in place.

GEN ZGC_Z_mul(GEN v, GEN n) given v a vector of ZG and $n \in Z$ return the vector (with the same type as v with entries $n \cdot v[i]$). Shallow function.

GEN G_ZGC_mul(GEN x, GEN v) given v a vector of ZG and $x \in G$ return the vector of $x \cdot v[i]$. Shallow function.

GEN ZGCs_add(GEN x, GEN y) add two sparse vectors of ZG elements (see Blackbox linear algebra below).

7.5.9 Blackbox linear algebra.

A sparse column zCs v is a t_COL with two components C and E which are t_VECSMALL of the same length, representing $\sum_i E[i] * e_{C[i]}$, where (e_j) is the canonical basis. A sparse matrix (zMs) is a t_VEC of zCs.

FpCs and FpMs are identical to the above, but $E[i]$ is now interpreted as a *signed* C long integer representing an element of \mathbf{F}_p . This is important since p can be so large that $p + E[i]$ would not fit in a C long.

RgCs and RgMs are similar, except that the type of the components of E is now unspecified. Functions handling those later objects must not depend on the type of those components.

It is not possible to derive the space dimension (number of rows) from the above data. Thus most functions take an argument nbrow which is the number of rows of the corresponding column/matrix in dense representation.

GEN zCs_to_ZC(GEN C, long nbrow) convert the sparse vector C to a dense ZC of dimension nbrow.

GEN zMs_to_ZM(GEN M, long nbrow) convert the sparse matrix M to a dense ZM whose columns have dimension nbrow.

GEN FpMs_FpC_mul(GEN M, GEN B, GEN p) multiply the sparse matrix M (over \mathbf{F}_p) by the FpC B . The result is an FpC, i.e. a dense vector.

GEN zMs_ZC_mul(GEN M, GEN B, GEN p) multiply the sparse matrix M by the ZC B (over \mathbf{Z}). The result is an ZC, i.e. a dense vector.

GEN FpV_FpMs_mul(GEN B, GEN M, GEN p) multiply the FpV B by the sparse matrix M (over \mathbf{F}_p). The result is an FpV, i.e. a dense vector.

GEN ZV_zMs_mul(GEN B, GEN M, GEN p) multiply the FpV B (over \mathbf{Z}) by the sparse matrix M . The result is an ZV, i.e. a dense vector.

void RgMs_structelim(GEN M, long nbrow, GEN A, GEN *p_col, GEN *p_row) M being a RgMs with nbrow rows, A being a list of row indices, Perform structured elimination on M by removing some rows and columns until the number of effectively present rows is equal to the number of columns. the result is stored in two t_VECSMALLs, *p_col and *p_row: *p_col is a map from the new columns indices to the old one. *p_row is a map from the old rows indices to the new one (0 if removed).

GEN FpMs_leftkernel_elt(GEN M, long nbrow, GEN p) M being a sparse matrix over \mathbf{F}_p , return a non-zero kbdFpV X such that XM components are almost all 0.

GEN FpMs_FpCs_solve(GEN M, GEN B, long nbrow, GEN p) solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over \mathbf{F}_p . Return either a solution as a t_COL

(dense vector), the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if B is not in the image of M).

`GEN FpMs_FpCs_solve_safe(GEN M, GEN B, long nbrow, GEN p)` as above, but in the event that p is not a prime and an impossible division occurs, return `NULL`.

`GEN ZpMs_ZpCs_solve(GEN M, GEN B, long nbrow, GEN p, long e)` solve the equation $MX = B$, where M is a sparse matrix and B is a sparse vector, both over $\mathbf{Z}/p^e\mathbf{Z}$. Return either a solution as a `t_COL` (dense vector), or the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component.

`GEN gen_FpM_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p)` solve the equation $f(X) = B$ over \mathbf{F}_p , where B is a `FpV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X . Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

`GEN gen_ZpM_Dixon(void *E, GEN (*f)(void*, GEN), GEN B, GEN p, long e)` solve equation $f(X) = B$ over $\mathbf{Z}/p^e\mathbf{Z}$, where B is a `ZV`, and f is a blackbox endomorphism, where $f(E, X)$ computes the value of f at the (dense) column vector X . Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

7.5.10 Obsolete functions.

The functions in this section are kept for backward compatibility only and will eventually disappear.

`GEN image2(GEN x)` compute the image of x using a very slow algorithm. Use `image` instead.

7.6 Integral, rational and generic polynomial arithmetic.

7.6.1 ZX.

`void RgX_check_ZX(GEN x, const char *s)` Assuming x is a `t_POL` raise an error if it is not a `ZX` (s should point to the name of the caller).

`GEN ZX_copy(GEN x, GEN p)` returns a copy of x .

`long ZX_max_lg(GEN x)` returns the effective length of the longest component in x .

`GEN scalar_ZX(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x .

`GEN scalar_ZX_shallow(GEN x, long v)` returns the constant `ZX` in variable v equal to the `t_INT` x . Shallow function not suitable for `gerepile` and friends.

`GEN ZX_renormalize(GEN x, long l)`, as `normalizepol`, where $l = \lg(x)$, in place.

`int ZX_equal(GEN x, GEN y)` returns 1 if the two `ZX` have the same `degpol` and their coefficients are equal. Variable numbers are not checked.

`int ZX_equal1(GEN x)` returns 1 if the `ZX` x is equal to 1 and 0 otherwise.

`int ZX_is_monic(GEN x)` returns 1 if the `ZX` x is monic and 0 otherwise. The zero polynomial considered not monic.

`GEN ZX_add(GEN x, GEN y)` adds x and y .

`GEN ZX_sub(GEN x, GEN y)` subtracts x and y .

GEN ZX_neg(GEN x) returns $-x$.

GEN ZX_Z_add(GEN x, GEN y) adds the integer y to the ZX x .

GEN ZX_Z_add_shallow(GEN x, GEN y) shallow version of ZX_Z_add.

GEN ZX_Z_sub(GEN x, GEN y) subtracts the integer y to the ZX x .

GEN Z_ZX_sub(GEN x, GEN y) subtracts the ZX y to the integer x .

GEN ZX_Z_mul(GEN x, GEN y) multiplies the ZX x by the integer y .

GEN ZX_mulu(GEN x, ulong y) multiplies x by the integer y .

GEN ZX_shifti(GEN x, long n) shifts all coefficients of x by n bits, which can be negative.

GEN ZX_Z_divexact(GEN x, GEN y) returns x/y assuming all divisions are exact.

GEN ZX_remi2n(GEN x, long n) reduces all coefficients of x to n bits, using remi2n.

GEN ZX_mul(GEN x, GEN y) multiplies x and y .

GEN ZX_sqr(GEN x, GEN p) returns x^2 .

GEN ZX_mulspec(GEN a, GEN b, long na, long nb). Internal routine: a and b are arrays of coefficients representing polynomials $\sum_{i=0}^{na-1} a[i]X^i$ and $\sum_{i=0}^{nb-1} b[i]X^i$. Returns their product (as a true GEN) in variable 0.

GEN ZX_sqrspec(GEN a, long na). Internal routine: a is an array of coefficients representing polynomial $\sum_{i=0}^{na-1} a[i]X^i$. Return its square (as a true GEN) in variable 0.

GEN ZX_rem(GEN x, GEN y) returns the remainder of the Euclidean division of x mod y . Assume that x, y are two ZX and that y is monic.

GEN ZX_mod_Xnm1(GEN T, ulong n) return T modulo $X^n - 1$. Shallow function.

GEN ZX_div_by_X_1(GEN T, GEN *r) return the quotient of T by $X - 1$. If r is not NULL set it to $T(1)$.

GEN ZX_gcd(GEN x, GEN y) returns a gcd of the ZX x and y . Not memory-clean, but suitable for gerepileupto.

GEN ZX_gcd_all(GEN x, GEN y, GEN *pX) returns a gcd d of x and y . If pX is not NULL, set $*pX$ to a (non-zero) integer multiple of x/d . If x and y are both monic, then d is monic and $*pX$ is exactly x/d . Not memory clean if the gcd is 1 (in that case $*pX$ is set to x).

GEN ZX_radical(GEN x) returns the largest squarefree divisor of the ZX x . Not memory clean.

GEN ZX_content(GEN x) returns the content of the ZX x .

long ZX_val(GEN P) as RgX_val, but assumes P has t_INT coefficients.

long ZX_valrem(GEN P, GEN *z) as RgX_valrem, but assumes P has t_INT coefficients.

GEN ZX_to_monic(GEN q GEN *L) given q a non-zero ZX, returns a monic integral polynomial Q such that $Q(x) = Cq(x/L)$, for some rational C and positive integer $L > 0$. If L is not NULL, set *L to L ; if $L = 1$, *L is set to gen_1. Not suitable for gerepileupto.

GEN ZX_primitive_to_monic(GEN q, GEN *L) as ZX_to_monic except q is assumed to have trivial content, which avoids recomputing it. The result is suboptimal if q is not primitive (L larger than necessary), but remains correct.

GEN ZX_Z_normalize(GEN q, GEN *L) a restricted version of ZX_primitive_to_monic, where q is a *monic* ZX of degree > 0 . Finds the largest integer $L > 0$ such that $Q(X) := L^{-\deg q}q(Lx)$ is integral and return Q ; this is not well-defined if q is a monomial, in that case, set $L = 1$ and $Q = q$. If L is not NULL, set $*L$ to L .

GEN ZX_Q_normalize(GEN q, GEN *L) a variant of ZX_Z_normalize where $L > 0$ is allowed to be rational, the monic $Q \in \mathbf{Z}[X]$ has possibly smaller coefficients.

GEN ZX_Q_mul(GEN x, GEN y) returns $x * y$, where y is a rational number and the resulting $\mathbf{t_POL}$ has rational entries.

long ZX_deflate_order(GEN P) given a non-constant ZX P , returns the largest exponent d such that P is of the form $P(x^d)$.

long ZX_deflate_max(GEN P, long *d). Given a non-constant polynomial with integer coefficients P , sets d to ZX_deflate_order(P) and returns RgX_deflate(P, d). Shallow function.

GEN ZX_rescale(GEN P, GEN h) returns $h^{\deg(P)}P(x/h)$. P is a ZX and h is a non-zero integer. Neither memory-clean nor suitable for gerepileupto.

GEN ZX_rescale2n(GEN P, long n) returns $2^{n \deg(P)}P(x \gg n)$ where P is a ZX. Neither memory-clean nor suitable for gerepileupto.

GEN ZX_rescale_1t(GEN P) returns the monic integral polynomial $h^{\deg(P)-1}P(x/h)$, where P is a non-zero ZX and h is its leading coefficient. Neither memory-clean nor suitable for gerepileupto.

GEN ZX_translate(GEN P, GEN c) assume P is a ZX and c an integer. Returns $P(X+c)$ (optimized for $c = \pm 1$).

GEN ZX_unscale(GEN P, GEN h) given a ZX P and a $\mathbf{t_INT}$ h , returns $P(hx)$. Not memory clean.

GEN ZX_z_unscale(GEN P, long h) given a ZX P , returns $P(hx)$. Not memory clean.

GEN ZX_unscale2n(GEN P, long n) given a ZX P , returns $P(x \ll n)$. Not memory clean.

GEN ZX_unscale_div(GEN P, GEN h) given a ZX P and a $\mathbf{t_INT}$ h such that $h \mid P(0)$, returns $P(hx)/h$. Not memory clean.

GEN ZX_eval1(GEN P) returns the integer $P(1)$.

GEN ZX_graeffe(GEN p) returns the Graeffe transform of p , i.e. the ZX q such that $p(x)p(-x) = q(x^2)$.

GEN ZX_deriv(GEN x) returns the derivative of x .

GEN ZX_resultant(GEN A, GEN B) returns the resultant of the ZX A and B .

GEN ZX_disc(GEN T) returns the discriminant of the ZX T .

GEN ZX_factor(GEN T) returns the factorization of the primitive part of T over $\mathbf{Q}[X]$ (the content is lost).

int ZX_is_squarefree(GEN T) returns 1 if the ZX T is squarefree, 0 otherwise.

long ZX_is_irred(GEN T) returns 1 if T is irreducible, and 0 otherwise.

GEN ZX_squff(GEN T, GEN *E) write T as a product $\prod T_i^{e_i}$ with the $e_1 < e_2 < \dots$ all distinct and the T_i pairwise coprime. Return the vector of the T_i , and set $*E$ to the vector of the e_i , as a $\mathbf{t_VECSMALL}$.

`GEN ZX_Uspensky(GEN P, GEN ab, long flag, long bitprec)` let P be a primitive ZX polynomial whose real roots are simple and `bitprec` is the relative precision in bits.

- If `flag` is 0 returns a list of intervals that isolate the real roots of P . The return value is a column of elements which are either vectors $[a, b]$ meaning that there is a single root in the open interval (a, b) or elements x_0 such that x_0 is a root of P . There is no guarantee that all rational roots are found (at most those with denominator a power of 2 can be found and even those are not guaranteed). Beware that the limits of the open intervals can be roots of the polynomial.

- If `flag` is 1 returns an approximation of the real roots of P .

- If `flag` is 2 returns the number of roots.

The argument `ab` specify the interval in which the roots are searched. The default interval is $(-\infty, \infty)$. If `ab` is an integer or fraction a then the interval is $[a, \infty)$. If `ab` is a vector $[a, b]$, where `t_INT`, `t_FRAC` or `t_INFINITY` are allowed for a and b , the interval is $[a, b]$.

`long ZX_sturm(GEN P)` number of real roots of the non-constant squarefree ZX P . For efficiency, it is advised to make P primitive first.

`long ZX_sturmpart(GEN P, GEN ab)` number of real roots of the non-constant squarefree ZX P in the interval specified by `ab`: either `NULL` (no restriction) or a `t_VEC` $[a, b]$ with two real components (of type `t_INT`, `t_FRAC` or `t_INFINITY`). For efficiency, it is advised to make P primitive first.

7.6.2 Resultants.

`GEN ZX_ZXY_resultant(GEN A, GEN B)` under the assumption that A in $\mathbf{Z}[Y]$, B in $\mathbf{Q}[Y][X]$, and $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$, returns the resultant R .

`GEN ZX_compositum_disjoint(GEN A, GEN B)` given two irreducible ZX defining linearly disjoint extensions, returns a ZX defining their compositum.

`GEN ZX_ZXY_rnfequation(GEN A, GEN B, long *lambda)`, assume A in $\mathbf{Z}[Y]$, B in $\mathbf{Q}[Y][X]$, and $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$. If `lambda` = `NULL`, returns R as in `ZY_ZXY_resultant`. Otherwise, `lambda` must point to some integer, e.g. 0 which is used as a seed. The function then finds a small $\lambda \in \mathbf{Z}$ (starting from `*lambda`) such that $R_\lambda(X) := \text{Res}_Y(A, B(X + \lambda Y))$ is squarefree, resets `*lambda` to the chosen value and returns R_λ .

7.6.3 ZXV.

`GEN ZXV_equal(GEN x, GEN y)` returns 1 if the two vectors of ZX are equal, as per `ZX_equal` (variables are not checked to be equal) and 0 otherwise.

`GEN ZXV_Z_mul(GEN x, GEN y)` multiplies the vector of ZX x by the integer y .

`GEN ZXV_remi2n(GEN x, long n)` applies `ZX_remi2n` to all coefficients of x .

`GEN ZXV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming x and y have ZX entries.

7.6.4 ZXT.

`GEN ZXT_remi2n(GEN x, long n)` applies `ZX_remi2n` to all leaves of the tree x .

7.6.5 ZXQ.

GEN ZXQ_mul(GEN x, GEN y, GEN T) returns $x * y \bmod T$, assuming that all inputs are ZXs and that T is monic.

GEN ZXQ_sqr(GEN x, GEN T) returns $x^2 \bmod T$, assuming that all inputs are ZXs and that T is monic.

GEN ZXQ_charpoly(GEN A, GEN T, long v): let T and A be ZXs, returns the characteristic polynomial of $\text{Mod}(A, T)$. More generally, A is allowed to be a QX, hence possibly has rational coefficients, *assuming* the result is a ZX, i.e. the algebraic number $\text{Mod}(A, T)$ is integral over Z .

7.6.6 ZXn.

GEN ZXn_mul(GEN x, GEN y, long n) return $xy \pmod{X^n}$.

GEN ZXn_sqr(GEN x, long n) return $x^2 \pmod{X^n}$.

GEN eta_ZXn(long r, long n) return $\eta(X^r) = \prod_{i>0} (1 - X^{ri}) \pmod{X^n}$, $r > 0$.

GEN eta_product_ZXn(GEN DR, long n): $DR = [D, R]$ being a vector with two $\mathfrak{t_VECSMALL}$ components, return $\prod_i \eta(X^{d_i})^{r_i}$. Shallow function.

7.6.7 ZXQM.

ZXQM are matrices of ZXQ. All entries must be integers or polynomials of degree strictly less than the degree of T .

GEN ZXQM_mul(GEN x, GEN y, GEN T) returns $x * y \bmod T$, assuming that all inputs are ZXs and that T is monic.

GEN ZXQM_sqr(GEN x, GEN T) returns $x^2 \bmod T$, assuming that all inputs are ZXs and that T is monic.

7.6.8 ZXQX.

GEN ZXQX_mul(GEN x, GEN y, GEN T) returns $x * y$, assuming that all inputs are ZXQXs and that T is monic.

GEN ZXQX_sqr(GEN x, GEN T) returns x^2 , assuming that all inputs are ZXQXs and that T is monic.

7.6.9 ZXX.

void RgX_check_ZXX(GEN x, const char *s) Assuming x is a $\mathfrak{t_POL}$ raise an error if it one of its coefficients is not an integer or a ZX (s should point to the name of the caller).

GEN ZXX_renormalize(GEN x, long l), as `normalizepol`, where $l = \lg(x)$, in place.

long ZXX_max_lg(GEN x) returns the effective length of the longest component in x ; assume all coefficients are $\mathfrak{t_INT}$ or ZXs.

GEN ZXX_Z_mul(GEN x, GEN y) returns xy .

GEN ZXX_Z_add_shallow(GEN x, GEN y) returns $x + y$. Shallow function.

GEN ZXX_Z_divexact(GEN x, GEN y) returns x/y assuming all integer divisions are exact.

GEN ZXX_to_Kronecker(GEN P, long n) Assuming $P(X, Y)$ is a polynomial of degree in X strictly less than n , returns $P(X, X^{2*n-1})$, the Kronecker form of P . Shallow function.

GEN ZXX_to_Kronecker_spec(GEN Q, long lQ, long n) return ZXX_to_Kronecker(P, n), where P is the polynomial $\sum_{i=0}^{lQ-1} Q[i]x^i$. To be used when splitting the coefficients of genuine polynomials into blocks. Shallow function.

GEN Kronecker_to_ZXX(GEN z, long n, long v) recover $P(X, Y)$ from its Kronecker form $P(X, X^{2n-1})$, v is the variable number corresponding to Y . Shallow function.

GEN ZXX_mul_Kronecker(GEN P, GEN Q, long n) return ZX_mul applied to the Kronecker forms $P(X, X^{2n-1})$ and $Q(X, X^{2n-1})$ of P and Q . Not memory clean.

GEN ZXX_sqr_Kronecker(GEN P, long n) return ZX_sqr applied to the Kronecker forms $P(X, X^{2n-1})$ of P . Not memory clean.

7.6.10 QX.

void RgX_check_QX(GEN x, const char *s) Assuming x is a `t_POL` raise an error if it is not a QX (s should point to the name of the caller).

GEN QX_mul(GEN x, GEN y)

GEN QX_sqr(GEN x)

GEN QX_ZX_rem(GEN x, GEN y) y is assumed to be monic.

GEN QX_gcd(GEN x, GEN y) returns a gcd of the QX x and y .

GEN QX_disc(GEN T) returns the discriminant of the QX T .

GEN QX_factor(GEN T) as ZX_factor.

GEN QX_resultant(GEN A, GEN B) returns the resultant of the QX A and B .

GEN QX_complex_roots(GEN p, long l) returns the complex roots of the QX p at accuracy l , where real roots are returned as `t_REALs`. More efficient when p is irreducible and primitive. Special case of `cleanroots`.

7.6.11 QXQ.

GEN QXQ_norm(GEN A, GEN B) A being a QX and B being a ZX, returns the norm of the algebraic number $A \bmod B$, using a modular algorithm. To ensure that B is a ZX, one may replace it by `Q_primpart(B)`, which of course does not change the norm.

If A is not a ZX — it has a denominator —, but the result is nevertheless known to be an integer, it is much more efficient to call `QXQ_intnorm` instead.

GEN QXQ_intnorm(GEN A, GEN B) A being a QX and B being a ZX, returns the norm of the algebraic number $A \bmod B$, *assuming* that the result is an integer, which is for instance the case is $A \bmod B$ is an algebraic integer, in particular if A is a ZX. To ensure that B is a ZX, one may replace it by `Q_primpart(B)` (which of course does not change the norm).

If the result is not known to be an integer, you must use `QXQ_norm` instead, which is slower.

GEN QXQ_mul(GEN A, GEN B, GEN T) returns the product of A and B modulo T where both A and B are a QX and T is a monic ZX.

GEN QXQ_sqr(GEN A, GEN T) returns the square of A modulo T where A is a QX and T is a monic ZX.

GEN QXQ_inv(GEN A, GEN B) returns the inverse of A modulo B where A is a QX and B is a ZX. Should you need this for a QX B , just use

```
QXQ_inv(A, Q_primpart(B));
```

But in all cases where modular arithmetic modulo B is desired, it is much more efficient to replace B by $Q_primpart(B)$ once and for all.

GEN QXQ_div_ratlift(GEN C, GEN A, GEN B) returns C/A modulo B where A and C are QX and B is a ZX. Use this function when the result is known to be “small” compared to $A^{-1} \bmod B$, it will be faster than QXQ_inv in this case.

GEN QXQ_charpoly(GEN A, GEN T, long v) where A is a QX and T is a ZX, returns the characteristic polynomial of $\text{Mod}(A, T)$. If the result is known to be a ZX, then calling ZXQ_charpoly will be faster.

GEN QXQ_powers(GEN x, long n, GEN T) returns $[x^0, \dots, x^n]$ as RgXQ_powers would, but in a more efficient way when x has a huge integer denominator (we start by removing that denominator). Meant to be used to precompute powers of algebraic integers in $\mathbf{Q}[t]/(T)$. The current implementation does not require x to be a QX: any polynomial to which Q_remove_denom can be applied is fine.

GEN QXQ_reverse(GEN f, GEN T) as RgXQ_reverse, assuming f is a QX.

GEN QX_ZXQV_eval(GEN f, GEN nV, GEN dV) as RgX_RgXQV_eval, except that f is assumed to be a QX, V is given implicitly by a numerator nV (ZV) and denominator dV (a positive t_INT or NULL for trivial denominator). Not memory clean, but suitable for gerepileupto.

GEN QXV_QXQ_eval(GEN v, GEN a, GEN T) v is a vector of QXs (possibly scalars, i.e. rational numbers, for convenience), a and T both QX. Return the vector of evaluations at a modulo T . Not memory clean, nor suitable for gerepileupto.

GEN QXX_QXQ_eval(GEN P, GEN a, GEN T) $P(X, Y)$ is a t_POL with QX coefficients (possibly scalars, i.e. rational numbers, for convenience), a and T both QX. Return the QX $P(X, a \bmod T)$. Not memory clean, nor suitable for gerepileupto.

GEN nfgcd(GEN P, GEN Q, GEN T, GEN den) given P and Q in $\mathbf{Z}[X, Y]$, T monic irreducible in $\mathbf{Z}[Y]$, returns the primitive d in $\mathbf{Z}[X, Y]$ which is a gcd of P, Q in $K[X]$, where K is the number field $\mathbf{Q}[Y]/(T)$. If not NULL, den is a multiple of the integral denominator of the (monic) gcd of P, Q in $K[X]$.

GEN nfgcd_all(GEN P, GEN Q, GEN T, GEN den, GEN *Pnew) as nfgcd. If Pnew is not NULL, set *Pnew to a non-zero integer multiple of P/d . If P and Q are both monic, then d is monic and *Pnew is exactly P/d . Not memory clean if the gcd is 1 (in that case *Pnew is set to P).

7.6.12 QXQM.

QXQM are matrices of QXQ. All entries must be t_INT, t_FRAC or polynomials of degree strictly less than the degree of T , which must be a monic ZX.

GEN QXQM_mul(GEN x, GEN y, GEN T) returns $x * y \bmod T$.

GEN QXQM_sqr(GEN x, GEN T) returns $x^2 \bmod T$.

7.6.13 zx.

GEN zero_zx(long sv) returns a zero zx in variable v .

GEN polx_zx(long sv) returns the variable v as degree 1 Flx.

GEN zx_renormalize(GEN x, long l), as Flx_renormalize, where $l = \lg(x)$, in place.

GEN zx_shift(GEN T, long n) returns T multiplied by x^n , assuming $n \geq 0$.

7.6.14 RgX.

7.6.14.1 Tests.

long RgX_degree(GEN x, long v) x being a t_POL and $v \geq 0$, returns the degree in v of x . Error if x is not a polynomial in v .

int RgX_isscalar(GEN x) return 1 if x all the coefficients of x of degree > 0 are 0 (as per gequal0).

int RgX_is_rational(GEN P) return 1 if the RgX P has only rational coefficients (t_INT and t_FRAC), and 0 otherwise.

int RgX_is_QX(GEN P) return 1 if the RgX P has only t_INT and t_FRAC coefficients, and 0 otherwise.

int RgX_is_ZX(GEN P) return 1 if the RgX P has only t_INT coefficients, and 0 otherwise.

int RgX_is_monomial(GEN x) returns 1 (true) if x is a non-zero monomial in its main variable, 0 otherwise.

long RgX_equal(GEN x, GEN y) returns 1 if the t_POLs x and y have the same degpol and their coefficients are equal (as per gequal). Variable numbers are not checked. Note that this is more stringent than gequal(x, y), which only checks whether $x - y$ satisfies gequal0; in particular, they may have different apparent degrees provided the extra leading terms are 0.

long RgX_equal_var(GEN x, GEN y) returns 1 if x and y have the same variable number and RgX_equal(x, y) is 1.

7.6.14.2 Coefficients, blocks.

GEN RgX_coeff(GEN P, long n) return the coefficient of x^n in P , defined as gen_0 if $n < 0$ or $n > \text{degpol}(P)$. Shallow function.

int RgX_blocks(GEN P, long n, long m) writes $P(X) = a_0(X) + X^n * a_1(X) * X^n + \dots + X^{n*(m-1)} a_{m-1}(X)$, where the a_i are polynomial of degree at most $n - 1$ (except possibly for the last one) and returns $[a_0(X), a_1(X), \dots, a_{m-1}(X)]$. Shallow function.

void RgX_even_odd(GEN p, GEN *pe, GEN *po) write $p(X) = E(X^2) + XO(X^2)$ and set *pe = E, *po = O. Shallow function.

GEN RgX_splitting(GEN P, long k) write $P(X) = a_0(X^k) + X a_1(X^k) + \dots + X^{k-1} a_{k-1}(X^k)$ and return $[a_0(X), a_1(X), \dots, a_{k-1}(X)]$. Shallow function.

GEN RgX_copy(GEN x) returns (a deep copy of) x .

GEN RgX_renormalize(GEN x) remove leading terms in x which are equal to (necessarily inexact) zeros.

GEN RgX_renormalize_lg(GEN x, long lx) as setlg(x, lx) followed by RgX_renormalize(x). Assumes that $lx \leq \lg(x)$.

GEN `RgX_recip`(GEN `P`) returns the reverse of the polynomial P , i.e. $X^{\deg P}P(1/X)$.

GEN `RgX_recip_shallow`(GEN `P`) shallow function of `RgX_recip`, where we further assume that $P(0) \neq 0$, so that the degree of the output is the degree of P .

GEN `RgX_deflate`(GEN `P`, long `d`) assuming P is a polynomial of the form $Q(X^d)$, return Q . Shallow function, not suitable for `gerepileupto`.

long `RgX_deflate_order`(GEN `P`) given a non-constant polynomial P , returns the largest exponent d such that P is of the form $P(x^d)$ (use `gequal0` to check whether coefficients are 0).

long `RgX_deflate_max`(GEN `P`, long `*d`) given a non-constant polynomial P , sets `d` to `RgX_deflate_order`(`P`) and returns `RgX_deflate`(`P`,`d`). Shallow function.

GEN `RgX_inflate`(GEN `P`, long `d`) return $P(X^d)$. Shallow function, not suitable for `gerepileupto`.

GEN `RgX_rescale_to_int`(GEN `x`) given a polynomial x with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a `ZX` which is very close to Dx for some well-chosen integer D . More precisely, if the input is exact, D is the denominator of x ; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

7.6.14.3 Shifts, valuations.

GEN `RgX_shift`(GEN `x`, long `n`) returns $x * t^n$ if $n \geq 0$, and $x \backslash t^{-n}$ otherwise.

GEN `RgX_shift_shallow`(GEN `x`, long `n`) as `RgX_shift`, but shallow (coefficients are not copied).

GEN `RgX_rotate_shallow`(GEN `P`, long `k`, long `p`) returns $P * X^k \pmod{X^p - 1}$, assuming the degree of P is strictly less than p , and $k \geq 0$.

void `RgX_shift_inplace_init`(long `v`) $v \geq 0$, prepare for a later call to `RgX_shift_inplace`. Reserves v words on the stack.

GEN `RgX_shift_inplace`(GEN `x`, long `v`) $v \geq 0$, assume that `RgX_shift_inplace_init`(v) has been called (reserving v words on the stack), immediately followed by a `t_POL` x . Return `RgX_shift`(x, v) by shifting x in place. To be used as follows

```
RgX_shift_inplace_init(v);
av = avma;
...
x = gerepileupto(av, ...); /* a t_POL */
return RgX_shift_inplace(x, v);
```

long `RgX_valrem`(GEN `P`, GEN `*pz`) returns the valuation v of the `t_POL` P with respect to its main variable X . Check whether coefficients are 0 using `isexactzero`. Set `*pz` to `RgX_shift_shallow`($P, -v$).

long `RgX_val`(GEN `P`) returns the valuation v of the `t_POL` P with respect to its main variable X . Check whether coefficients are 0 using `isexactzero`.

long `RgX_valrem_inexact`(GEN `P`, GEN `*z`) as `RgX_valrem`, using `gequal0` instead of `isexactzero`.

7.6.14.4 Basic arithmetic.

GEN `RgX_add`(GEN `x`, GEN `y`) adds `x` and `y`.

GEN `RgX_sub`(GEN `x`, GEN `y`) subtracts `x` and `y`.

GEN `RgX_neg`(GEN `x`) returns $-x$.

GEN `RgX_Rg_add`(GEN `y`, GEN `x`) returns $x + y$.

GEN `RgX_Rg_add_shallow`(GEN `y`, GEN `x`) returns $x + y$; shallow function.

GEN `Rg_RgX_sub`(GEN `x`, GEN `y`)

GEN `RgX_Rg_sub`(GEN `y`, GEN `x`) returns $x - y$

GEN `RgX_Rg_mul`(GEN `y`, GEN `x`) multiplies the RgX `y` by the scalar `x`.

GEN `RgX_muls`(GEN `y`, long `s`) multiplies the RgX `y` by the long `s`.

GEN `RgX_Rg_div`(GEN `y`, GEN `x`) divides the RgX `y` by the scalar `x`.

GEN `RgX_divs`(GEN `y`, long `s`) divides the RgX `y` by the long `s`.

GEN `RgX_Rg_divexact`(GEN `x`, GEN `y`) exact division of the RgX `y` by the scalar `x`.

GEN `RgX_Rg_eval_bk`(GEN `f`, GEN `x`) returns $f(x)$ using Brent and Kung algorithm. (Use `poleval` for Horner algorithm.)

GEN `RgX_RgV_eval`(GEN `f`, GEN `V`) as `RgX_Rg_eval_bk(f, x)`, assuming `V` was output by `gpowers(x, n)` for some $n \geq 1$.

GEN `RgXV_RgV_eval`(GEN `f`, GEN `V`) apply `RgX_RgV_eval_bk(, V)` to all the components of the vector `f`.

GEN `RgX_normalize`(GEN `x`) divides x by its leading coefficient. If the latter is 1, x itself is returned, not a copy. Leading coefficients equal to 0 are stripped, e.g.

$$0.*t^3 + \text{Mod}(0,3)*t^2 + 2*t$$

is normalized to t .

GEN `RgX_mul`(GEN `x`, GEN `y`) multiplies the two `t_POL` (in the same variable) `x` and `y`. Detect the coefficient ring and use an appropriate algorithm.

GEN `RgX_mul_i`(GEN `x`, GEN `y`) multiplies the two `t_POL` (in the same variable) `x` and `y`. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

GEN `RgX_mul_normalized`(GEN `A`, long `a`, GEN `B`, long `b`) returns $(X^a + A)(X^b + B) - X^{(a+b)}$, where we assume that $\deg A < a$ and $\deg B < b$ are polynomials in the same variable X .

GEN `RgX_sqr`(GEN `x`) squares the `t_POL` `x`. Detect the coefficient ring and use an appropriate algorithm.

GEN `RgX_sqr_i`(GEN `x`) squares the `t_POL` `x`. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

GEN `RgX_divrem`(GEN `x`, GEN `y`, GEN `*r`) by default, returns the Euclidean quotient and store the remainder in `r`. Three special values of `r` change that behavior • `NULL`: do not store the remainder, used to implement `RgX_div`,

- `ONLY_REM`: return the remainder, used to implement `RgX_rem`,

- **ONLY_DIVIDES**: return the quotient if the division is exact, and **NULL** otherwise.

GEN RgX_div(**GEN** x, **GEN** y)

GEN RgX_div_by_X_x(**GEN** A, **GEN** a, **GEN** *r) returns the quotient of the RgX A by $(X - a)$, and sets r to the remainder A(a).

GEN RgX_rem(**GEN** x, **GEN** y)

GEN RgX_pseudodivrem(**GEN** x, **GEN** y, **GEN** *ptr) compute a pseudo-quotient q and pseudo-remainder r such that $\text{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$. Return q and set *ptr to r .

GEN RgX_pseudorem(**GEN** x, **GEN** y) return the remainder in the pseudo-division of x by y .

GEN RgXQX_pseudorem(**GEN** x, **GEN** y, **GEN** T) return the remainder in the pseudo-division of x by y over $R[X]/(T)$.

int ZXQX_dvd(**GEN** x, **GEN** y, **GEN** T) let T be a monic irreducible ZX, let x, y be **t_POL** whose coefficients are either **t_INTs** or **ZX** in the same variable as T . Assume further that the leading coefficient of y is an integer. Return 1 if $y|x$ in $(\mathbf{Z}[Y]/(T))[X]$, and 0 otherwise.

GEN RgXQX_pseudodivrem(**GEN** x, **GEN** y, **GEN** T, **GEN** *ptr) compute a pseudo-quotient q and pseudo-remainder r such that $\text{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$ in $R[X]/(T)$. Return q and set *ptr to r .

GEN RgX_mulXn(**GEN** a, **long** n) returns $a * X^n$. This may be a **t_FRAC** if $n < 0$ and the valuation of a is not large enough.

GEN RgX_addmulXn(**GEN** a, **GEN** b, **long** n) returns $a + b * X^n$, assuming that $n > 0$.

GEN RgX_addmulXn_shallow(**GEN** a, **GEN** b, **long** n) shallow variant of RgX_addmulXn.

GEN RgX_digits(**GEN** x, **GEN** B) returns a vector of RgX $[c_0, \dots, c_n]$ of degree less than the degree of B and such that $x = \sum_{i=0}^n c_i B^i$.

7.6.14.5 Internal routines working on coefficient arrays.

These routines operate on coefficient blocks which are invalid **GENs** A **GEN** argument a or b in routines below is actually a coefficient arrays representing the polynomials $\sum_{i=0}^{na-1} a[i]X^i$ and $\sum_{i=0}^{nb-1} b[i]X^i$. Note that $a[0]$ and $b[0]$ contain coefficients and not the mandatory **GEN** codeword. This allows to implement divide-and-conquer methods directly, without needing to allocate wrappers around coefficient blocks.

GEN RgX_mulspec(**GEN** a, **GEN** b, **long** na, **long** nb). Internal routine: given two coefficient arrays representing polynomials, return their product (as a true **GEN**) in variable 0.

GEN RgX_sqrspec(**GEN** a, **long** na). Internal routine: given a coefficient array representing a polynomial r return its square (as a true **GEN**) in variable 0.

GEN RgX_addspec(**GEN** x, **GEN** y, **long** nx, **long** ny) given two coefficient arrays representing polynomials, return their sum (as a true **GEN**) in variable 0.

GEN RgX_addspec_shallow(**GEN** x, **GEN** y, **long** nx, **long** ny) shallow variant of RgX_addspec.

7.6.14.6 GCD, Resultant.

GEN `RgX_gcd`(GEN `x`, GEN `y`) returns the GCD of `x` and `y`, assumed to be `t_POLs` in the same variable.

GEN `RgX_gcd_simple`(GEN `x`, GEN `y`) as `RgX_gcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_gcd`.

GEN `RgX_extgcd`(GEN `x`, GEN `y`, GEN `*u`, GEN `*v`) returns $d = \text{GCD}(x, y)$, and sets `*u`, `*v` to the Bezout coefficients such that $*ux + *vy = d$. Uses a generic subresultant algorithm.

GEN `RgX_extgcd_simple`(GEN `x`, GEN `y`, GEN `*u`, GEN `*v`) as `RgX_extgcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_extgcd`.

GEN `RgX_disc`(GEN `x`) returns the discriminant of the `t_POL` `x` with respect to its main variable.

GEN `RgX_resultant_all`(GEN `x`, GEN `y`, GEN `*sol`) returns `resultant(x,y)`. If `sol` is not NULL, sets it to the last non-constant remainder in the polynomial remainder sequence if it exists and to `gen_0` otherwise (e.g. one polynomial has degree 0).

7.6.14.7 Other operations.

GEN `RgX_gtofp`(GEN `x`, GEN `prec`) returns the polynomial obtained by applying

`gtofp(gel(x,i), prec)`

to all coefficients of `x`.

GEN `RgX_fpnorml2`(GEN `x`, long `prec`) returns (a stack-clean variant of)

`gnorml2(RgX_gtofp(x, prec))`

GEN `RgX_deriv`(GEN `x`) returns the derivative of `x` with respect to its main variable.

GEN `RgX_integ`(GEN `x`) returns the primitive of `x` vanishing at 0, with respect to its main variable.

GEN `RgX_rescale`(GEN `P`, GEN `h`) returns $h^{\deg(P)}P(x/h)$. `P` is an `RgX` and `h` is non-zero. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

GEN `RgX_unscale`(GEN `P`, GEN `h`) returns $P(hx)$. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

GEN `RgXV_unscale`(GEN `v`, GEN `h`) apply `RgX_unscale` to a vector of `RgX`.

GEN `RgX_translate`(GEN `P`, GEN `c`) assume `c` is a scalar or a polynomials whose main variable has lower priority than the main variable `X` of `P`. Returns $P(X + c)$ (optimized for $c = \pm 1$).

7.6.14.8 Function related to modular forms.

GEN `RgX_act_Gl2Q`(GEN `g`, long `k`) let R be a commutative ring and $g = [a, b; c, d]$ be in $\text{GL}_2(\mathbf{Q})$, g acts (on the left) on homogeneous polynomials of degree $k - 2$ in $V := R[X, Y]_{k-2}$ via

$$g \cdot P := P(dX - cY, -bX + aY) = (\det g)^{k-2} P((X, Y) \cdot g^{-1}).$$

This function returns the matrix in $M_{k-1}(R)$ of $P \mapsto g \cdot P$ in the basis $(X^{k-2}, \dots, Y^{k-2})$ of V .

GEN `RgX_act_ZGl2Q`(GEN `z`, long `k`) let $G := \text{GL}_2(\mathbf{Q})$, acting on $R[X, Y]_{k-2}$ and $z \in \mathbf{Z}[G]$. Return the matrix giving $P \mapsto z \cdot P$ in the basis $(X^{k-2}, \dots, Y^{k-2})$.

7.6.15 RgXn.

GEN RgXn_red_shallow(GEN x, long n) return $x \% t^n$, where $n \geq 0$. Shallow function.

GEN RgXn_recip_shallow(GEN P) returns $X^n P(1/X)$. Shallow function.

GEN RgXn_mul(GEN a, GEN b, long n) returns ab modulo X^n , where a, b are two t_POL in the same variable X and $n \geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

GEN RgXn_sqr(GEN a, long n) returns a^2 modulo X^n , where a is a t_POL in the variable X and $n \geq 0$. Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

GEN RgX_mulhigh_i(GEN f, GEN g, long n) return the Euclidean quotient of $f(x) * g(x)$ by x^n (high product). Uses RgXn_mul applied to the reciprocal polynomials of f and g . Not suitable for gerepile.

GEN RgX_sqrhigh_i(GEN f, long n) return the Euclidean quotient of $f(x)^2$ by x^n (high product). Uses RgXn_sqr applied to the reciprocal polynomial of f . Not suitable for gerepile.

GEN RgXn_inv(GEN a, long n) returns a^{-1} modulo X^n , where a is a t_POL in the variable X and $n \geq 0$. Uses Newton-Raphson algorithm.

GEN RgXn_inv_i(GEN a, long n) as RgXn_inv without final garbage collection (suitable for gerepileupto).

GEN RgXn_powers(GEN x, long m, long n) returns $[x^0, \dots, x^m]$ modulo X^n as a t_VEC of RgXns.

GEN RgXn_powu(GEN x, ulong m, long n) returns x^m modulo X^n .

GEN RgXn_powu_i(GEN x, ulong m, long n) as RgXn_powu, not memory clean.

GEN RgXn_sqrt(GEN a, long n) returns $a^{1/2}$ modulo X^n , where a is a t_POL in the variable X and $n \geq 0$. Assume that $a = 1 \pmod{X}$. Uses Newton algorithm.

GEN RgXn_exp(GEN a, long n) returns $exp(a)$ modulo X^n , assuming $a = 0 \pmod{X}$. Uses Hanrot-Zimmermann algorithm.

GEN RgXn_eval(GEN Q, GEN x, long n) special case of RgX_RgXQ_eval, when the modulus is a monomial: returns $Q(x)$ modulo t^n , where $x \in R[t]$.

GEN RgX_RgXn_eval(GEN f, GEN x, long n) returns $f(x)$ modulo X^n .

GEN RgX_RgXnV_eval(GEN f, GEN V, long n) as RgX_RgXn_eval(f, x, n), assuming V was output by RgXn_powers(x, m, n) for some $m \geq 1$.

GEN RgXn_reverse(GEN f, long n) assuming that $f = ax \pmod{x^2}$ with a invertible, returns a t_POL g of degree $< n$ such that $(g \circ f)(x) = x \pmod{x^n}$.

7.6.16 RgXnV.

GEN RgXnV_red_shallow(GEN x, long n) apply RgXn_red_shallow to all the components of the vector x .

7.6.17 RgXQ.

GEN RgXQ_mul(GEN y, GEN x, GEN T) computes $xy \bmod T$

GEN RgXQ_sqr(GEN x, GEN T) computes $x^2 \bmod T$

GEN RgXQ_inv(GEN x, GEN T) return the inverse of $x \bmod T$.

GEN RgXQ_pow(GEN x, GEN n, GEN T) computes $x^n \bmod T$

GEN RgXQ_powu(GEN x, ulong n, GEN T) computes $x^n \bmod T$, n being an ulong.

GEN RgXQ_powers(GEN x, long n, GEN T) returns $[x^0, \dots, x^n]$ as a t_VEC of RgXQs.

GEN RgXQ_matrix_pow(GEN y, long n, long m, GEN P) returns RgXQ_powers($y, m-1, P$), as a matrix of dimension $n \geq \deg P$.

GEN RgXQ_norm(GEN x, GEN T) returns the norm of $\text{Mod}(x, T)$.

GEN RgXQ_charpoly(GEN x, GEN T, long v) returns the characteristic polynomial of $\text{Mod}(x, T)$, in variable v .

GEN RgX_RgXQ_eval(GEN f, GEN x, GEN T) returns $f(x)$ modulo T .

GEN RgX_RgXQV_eval(GEN f, GEN V, GEN T) as RgX_RgXQ_eval(f, x, T), assuming V was output by RgXQ_powers(x, n, T) for some $n \geq 1$.

int RgXQ_ratlift(GEN x, GEN T, long amax, long bmax, GEN *P, GEN *Q) Assuming that $\text{amax} + \text{bmax} < \deg T$, attempts to recognize x as a rational function a/b , i.e. to find t_POLs P and Q such that

- $P \equiv Qx \bmod T$,
- $\deg P \leq \text{amax}$, $\deg Q \leq \text{bmax}$,
- $\gcd(T, P) = \gcd(P, Q)$.

If unsuccessful, the routine returns 0 and leaves P, Q unchanged; otherwise it returns 1 and sets P and Q .

GEN RgXQ_reverse(GEN f, GEN T) returns a t_POL g of degree $< n = \deg T$ such that $T(x)$ divides $(g \circ f)(x) - x$, by solving a linear system. Low-level function underlying `modreverse`: it returns a lift of `[modreverse(f,T)]`; faster than the high-level function since it needs not compute the characteristic polynomial of $f \bmod T$ (often already known in applications). In the trivial case where $n \leq 1$, returns a scalar, not a constant t_POL.

7.6.18 RgXQV, RgXQC.

GEN RgXQC_red(GEN z, GEN T) z a vector whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying `grem` coefficientwise) in a t_COL.

GEN RgXQV_red(GEN z, GEN T) z a vector whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying `grem` coefficientwise) in a t_VEC.

GEN RgXQV_RgXQ_mul(GEN z, GEN x, GEN T) z multiplies the RgXQV z by the scalar (RgXQ) x .

7.6.19 RgXQM.

GEN RgXQM_red(GEN z, GEN T) z a matrix whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise).

GEN RgXQM_mul(GEN x, GEN y, GEN T)

7.6.20 RgXQX.

GEN RgXQX_red(GEN z, GEN T) z a t_POL whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise).

GEN RgXQX_mul(GEN x, GEN y, GEN T)

GEN RgXQX_RgXQ_mul(GEN x, GEN y, GEN T) multiplies the RgXQX y by the scalar (RgXQ) x.

GEN RgXQX_sqr(GEN x, GEN T)

GEN RgXQX_powers(GEN x, long n, GEN T)

GEN RgXQX_divrem(GEN x, GEN y, GEN T, GEN *pr)

GEN RgXQX_div(GEN x, GEN y, GEN T, GEN *r)

GEN RgXQX_rem(GEN x, GEN y, GEN T, GEN *r)

GEN RgXQX_translate(GEN P, GEN c, GEN T) assume the main variable X of P has higher priority than the main variable Y of T and c . Return a lift of $P(X + \text{Mod}(c(Y), T(Y)))$.

GEN Kronecker_to_mod(GEN z, GEN T) $z \in R[X]$ represents an element $P(X, Y)$ in $R[X, Y] \bmod T(Y)$ in Kronecker form, i.e. $z = P(X, X^{2*n-1})$

Let R be some commutative ring, $n = \deg T$ and let $P(X, Y) \in R[X, Y]$ lift a polynomial in $K[Y]$, where $K := R[X]/(T)$ and $\deg_X P < 2n - 1$ — such as would result from multiplying minimal degree lifts of two polynomials in $K[Y]$. Let $z = P(t, t^{2*n-1})$ be a Kronecker form of P , this function returns the image of $P(X, t)$ in $K[t]$, with t_POLMOD coefficients. Not stack-clean. Note that t need not be the same variable as Y !

Chapter 8: Black box algebraic structures

The generic routines like `gmul` or `gadd` allow handling objects belonging to a fixed list of basic types, with some natural polymorphism (you can mix rational numbers and polynomials, etc.), at the expense of efficiency and sometimes of clarity when the recursive structure becomes complicated, e.g. a few levels of `t_POLMODs` attached to different polynomials and variable numbers for quotient structures. This is the only possibility in GP.

On the other hand, the Level 2 Kernel allows dedicated routines to handle efficiently objects of a very specific type, e.g. polynomials with coefficients in the same finite field. This is more efficient, but involves a lot of code duplication since polymorphism is no longer possible.

A third and final option, still restricted to library programming, is to define an arbitrary algebraic structure (currently groups, fields, rings, algebras and \mathbf{Z}_p -modules) by providing suitable methods, then using generic algorithms. For instance naive Gaussian pivoting applies over all base fields and need only be implemented once. The difference with the first solution is that we no longer depend on the way functions like `gmul` or `gadd` will guess what the user is trying to do. We can then implement independently various groups / fields / algebras in a clean way.

8.1 Black box groups.

A black box group is defined by a `bb_group` struct, describing methods available to handle group elements:

```
struct bb_group
{
    GEN (*mul)(void*, GEN, GEN);
    GEN (*pow)(void*, GEN, GEN);
    GEN (*rand)(void*);
    ulong (*hash)(GEN);
    int (*equal)(GEN, GEN);
    int (*equal1)(GEN);
    GEN (*easylog)(void *E, GEN, GEN, GEN);
};
```

`mul(E,x,y)` returns the product xy .

`pow(E,x,n)` returns x^n (n integer, possibly negative or zero).

`rand(E)` returns a random element in the group.

`hash(x)` returns a hash value for x (`hash_GEN` is suitable for this field).

`equal(x,y)` returns one if $x = y$ and zero otherwise.

`equal1(x)` returns one if x is the neutral element in the group, and zero otherwise.

`easylog(E,a,g,o)` (optional) returns either `NULL` or the discrete logarithm n such that $g^n = a$, the element g being of order o . This provides a short-cut in situation where a better algorithm than the generic one is known.

A group is thus described by a `struct bb_group` as above and auxiliary data typecast to `void*`. The following functions operate on black box groups:

`GEN gen_Shanks_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)`
 Generic baby-step/giant-step algorithm (Shanks's method). Assuming that g has order N , compute an integer k such that $g^k = x$. Return `cgetg(1, t_VEC)` if there are no solutions. This requires $O(\sqrt{N})$ group operations and uses an auxiliary table containing $O(\sqrt{N})$ group elements.

The above is useful for a one-shot computation. If many discrete logs are desired: `GEN gen_Shanks_init(GEN g, long n, void *E, const struct bb_group *grp)` return an auxiliary data structure T required to compute a discrete log in base g . Compute and store all powers g^i , $i < n$.

`GEN gen_Shanks(GEN T, GEN x, ulong N, void *E, const struct bb_group *grp)` Let T be computed by `gen_Shanks_init(g, n, ...)`. Return $k < nN$ such that $g^k = x$ or `NULL` if no such index exist. It uses $O(N)$ operation in the group and fast table lookups (in time $O(\log n)$). The interface is such that the function may be used when the order of the base g is unknown, and hence compute it given only an upper bound B for it: e.g. choose n, N such that $nN \geq B$ and compute the discrete log l of g^{-1} in base g , then use `gen_order` with multiple $N = l + 1$.

`GEN gen_Pollard_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)`
 Generic Pollard rho algorithm. Assuming that g has order N , compute an integer k such that $g^k = x$. This requires $O(\sqrt{N})$ group operations in average and $O(1)$ storage. Will enter an infinite loop if there are no solutions.

`GEN gen_plog(GEN x, GEN g, GEN N, void *E, const struct bb_group)` Assuming that g has prime order N , compute an integer k such that $g^k = x$, using either `gen_Shanks_log` or `gen_Pollard_log`. Return `cgetg(1, t_VEC)` if there are no solutions.

`GEN gen_Shanks_sqrtn(GEN a, GEN n, GEN N, GEN *zetan, void *E, const struct bb_group *grp)` returns one solution of $x^n = a$ in a black box cyclic group of order N . Return `NULL` if no solution exists. If `zetan` is not `NULL` it is set to an element of exact order n . This function uses `gen_plog` for all prime divisors of $\gcd(n, N)$.

`GEN gen_PH_log(GEN a, GEN g, GEN N, void *E, const struct bb_group *grp)` returns an integer k such that $g^k = x$, assuming that g has order N , by Pohlig-Hellman algorithm. Return `cgetg(1, t_VEC)` if there are no solutions. This calls `gen_plog` repeatedly for all prime divisors p of N .

In the following functions the integer parameter `ord` can be given in all the formats recognized for the argument of arithmetic functions, i.e. either as a positive `t_INT` N , or as its factorization matrix faN , or (preferred) as a pair $[N, faN]$.

`GEN gen_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` computes the order of x ; `ord` is a multiple of the order, for instance the group order.

`GEN gen_factored_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` returns a pair $[o, F]$, where o is the order of x and F is the factorization of o ; `ord` is as in `gen_order`.

`GEN gen_gener(GEN ord, void *E, const struct bb_group *grp)` returns a random generator of the group, assuming it is of order exactly `ord`.

`GEN get_arith_Z(GEN ord)` given `ord` as above in one of the formats recognized for arithmetic functions, i.e. a positive `t_INT` N , its factorization faN , or the pair $[N, faN]$, return N .

GEN `gen_arith_ZZM`(GEN `ord`) given `ord` as above, return the pair $[N, faN]$. This may require factoring N .

GEN `gen_select_order`(GEN `v`, void `*E`, const struct `bb_group *grp`) Let v be a vector of possible orders for the group; try to find the true order by checking orders of random points. This will not terminate if there is an ambiguity.

8.1.1 Black box groups with pairing.

These functions handle groups of rank at most 2 equipped with a family of bilinear pairings which behave like the Weil pairing on elliptic curves over finite field. In the descriptions below, the function `pairorder`(`E`, `P`, `Q`, `m`, `F`) must return the order of the m -pairing of P and Q , both of order dividing m , where F is the factorization matrix of a multiple of m .

GEN `gen_ellgroup`(GEN `o`, GEN `d`, GEN `*pt_m`, void `*E`, const struct `bb_group *grp`, GEN `pairorder`(void `*E`, GEN `P`, GEN `Q`, GEN `m`, GEN `F`)) returns the elementary divisors $[d_1, d_2]$ of the group, assuming it is of order exactly $o > 1$, and that d_2 divides d . If $d_2 = 1$ then $[o]$ is returned, otherwise `m=*pt_m` is set to the order of the pairing required to verify a generating set which is to be used with `gen_ellgens`. For the parameter o , all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair $[n, \text{factor}(n)]$.

GEN `gen_ellgens`(GEN `d1`, GEN `d2`, GEN `m`, void `*E`, const struct `bb_group *grp`, GEN `pairorder`(void `*E`, GEN `P`, GEN `Q`, GEN `m`, GEN `F`)) the parameters d_1, d_2, m being as returned by `gen_ellgroup`, returns a pair of generators $[P, Q]$ such that P is of order d_1 and the m -pairing of P and Q is of order m . (Note: Q needs not be of order d_2). For the parameter d_1 , all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair $[n, \text{factor}(n)]$.

8.1.2 Functions returning black box groups.

const struct `bb_group * get_Flxq_star`(void `**E`, GEN `T`, ulong `p`)

const struct `bb_group * get_FpXQ_star`(void `**E`, GEN `T`, GEN `p`) returns a pointer to the black box group $(\mathbf{F}_p[x]/(T))^*$.

const struct `bb_group * get_FpE_group`(void `**pE`, GEN `a4`, GEN `a6`, GEN `p`) returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group $E(\mathbf{F}_p)$ where E is the elliptic curve $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in \mathbf{F}_p .

const struct `bb_group * get_FpXQE_group`(void `**pE`, GEN `a4`, GEN `a6`, GEN `T`, GEN `p`) returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group $E(\mathbf{F}_p[X]/(T))$ where E is the elliptic curve $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$.

const struct `bb_group * get_FlxqE_group`(void `**pE`, GEN `a4`, GEN `a6`, GEN `T`, ulong `p`) idem for small p .

const struct `bb_group * get_F2xqE_group`(void `**pE`, GEN `a2`, GEN `a6`, GEN `T`) idem for $p = 2$.

8.2 Black box fields.

A black box field is defined by a `bb_field` struct, describing methods available to handle field elements:

```
struct bb_field
{
    GEN (*red)(void *E ,GEN);
    GEN (*add)(void *E ,GEN, GEN);
    GEN (*mul)(void *E ,GEN, GEN);
    GEN (*neg)(void *E ,GEN);
    GEN (*inv)(void *E ,GEN);
    int (*equal0)(GEN);
    GEN (*s)(void *E, long);
};
```

In contrast of black box group, elements can have non canonical forms, and only `red` is required to return a canonical form. For instance a black box implementation of finite fields, all methods except `red` may return arbitrary representatives in $\mathbf{Z}[X]$ of the correct congruence class modulo $(p, T(X))$.

`red(E,x)` returns the canonical form of x .

`add(E,x,y)` returns the sum $x + y$.

`mul(E,x,y)` returns the product xy .

`neg(E,x)` returns $-x$.

`inv(E,x)` returns the inverse of x .

`equal0(x)` x being in canonical form, returns one if $x = 0$ and zero otherwise.

`s(n)` n being a small signed integer, returns n times the unit element.

A field is thus described by a `struct bb_field` as above and auxiliary data typecast to `void*`. The following functions operate on black box fields:

```
GEN gen_Gauss(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_Gauss_pivot(GEN x, long *rr, void *E, const struct bb_field *ff)
GEN gen_det(GEN a, void *E, const struct bb_field *ff)
GEN gen_ker(GEN x, long deplin, void *E, const struct bb_field *ff)
GEN gen_matcolinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matcolmul(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matid(long n, void *E, const struct bb_field *ff)
GEN gen_matinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matmul(GEN a, GEN b, void *E, const struct bb_field *ff)
```

8.2.1 Functions returning black box fields.

```
const struct bb_field * get_Fp_field(void **pE, GEN p)
const struct bb_field * get_Fq_field(void **pE, GEN T, GEN p)
const struct bb_field * get_Flxq_field(void **pE, GEN T, ulong p)
const struct bb_field * get_F2xq_field(void **pE, GEN T)
const struct bb_field * get_nf_field(void **pE, GEN nf)
```

8.3 Black box algebra.

A black box algebra is defined by a `bb_algebra` struct, describing methods available to handle algebra elements:

```
struct bb_algebra
{
    GEN (*red)(void *E, GEN x);
    GEN (*add)(void *E, GEN x, GEN y);
    GEN (*sub)(void *E, GEN x, GEN y);
    GEN (*mul)(void *E, GEN x, GEN y);
    GEN (*sqr)(void *E, GEN x);
    GEN (*one)(void *E);
    GEN (*zero)(void *E);
};
```

In contrast with black box groups, elements can have non canonical forms, but only `add` is allowed to return a non canonical form.

`red(E,x)` returns the canonical form of x .

`add(E,x,y)` returns the sum $x + y$.

`sub(E,x,y)` returns the difference $x - y$.

`mul(E,x,y)` returns the product xy .

`sqr(E,x)` returns the square x^2 .

`one(E)` returns the unit element.

`zero(E)` returns the zero element.

An algebra is thus described by a `struct bb_algebra` as above and auxiliary data typecast to `void*`. The following functions operate on black box algebra:

`GEN gen_bkeval(GEN P, long d, GEN x, int use_sqr, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x))` x being an element of the black box algebra, and P some black box polynomial of degree d over the base field, returns $P(x)$. The function `cmul(E,P,a,y)` must return the coefficient of degree a of P multiplied by y . `cmul` is allowed to return a non canonical form; it is also allowed to return `NULL` instead of an exact 0.

The flag `use_sqr` has the same meaning as for `gen_powers`. This implements an algorithm of Brent and Kung (1978).

GEN gen_bkeval_powers(GEN P, long d, GEN V, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x)) as gen_RgX_bkeval assuming V was output by gen_powers(x, l, E, ff) for some $l \geq 1$. For optimal performance, l should be computed by brent_kung_optpow.

long brent_kung_optpow(long d, long n, long m) returns the optimal parameter l for the evaluation of n/m polynomials of degree d . Fractional values can be used if the evaluations are done with different accuracies, and thus have different weights.

8.3.1 Functions returning black box algebras.

const struct bb_algebra * get_FpX_algebra(void **E, GEN p, long v) return the algebra of polynomials over \mathbf{F}_p in variable v .

const struct bb_algebra * get_FpXQ_algebra(void **E, GEN T, GEN p) return the algebra $\mathbf{F}_p[X]/(T(X))$.

const struct bb_algebra * get_FpXQX_algebra(void **E, GEN T, GEN p, long v) return the algebra of polynomials over $\mathbf{F}_p[X]/(T(X))$ in variable v .

const struct bb_algebra * get_FlxqXQ_algebra(void **E, GEN S, GEN T, ulong p) return the algebra $\mathbf{F}_p[X, Y]/(S(X, Y), T(X))$ (for ulong p).

const struct bb_algebra * get_FpXQXQ_algebra(void **E, GEN S, GEN T, GEN p) return the algebra $\mathbf{F}_p[X, Y]/(S(X, Y), T(X))$.

const struct bb_algebra * get_Rg_algebra(void) return the generic algebra.

8.4 Black box ring.

A black box ring is defined by a bb_ring struct, describing methods available to handle ring elements:

```
struct bb_ring
{
  GEN (*add)(void *E, GEN x, GEN y);
  GEN (*mul)(void *E, GEN x, GEN y);
  GEN (*sqr)(void *E, GEN x);
};
```

add(E,x,y) returns the sum $x + y$.

mul(E,x,y) returns the product xy .

sqr(E,x) returns the square x^2 .

GEN gen_fromdigits(GEN v, GEN B, void *E, struct bb_ring *r) where B is a ring element and $v = [c_0, \dots, c_{n-1}]$ a vector of ring elements, return $\sum_{i=0}^n c_i B^i$ using binary splitting.

GEN gen_digits(GEN x, GEN B, long n, void *E, struct bb_ring *r, GEN (*div)(void *E, GEN x, GEN y, GEN *r))

(Require the ring to be Euclidean)

div(E,x,y,&r) performs the Euclidean division of x by y in the ring R , returning the quotient q and setting r to the residue so that $x = qy + r$ holds. The residue must belong to a fixed set of representatives of $R/(y)$.

The argument x being a ring element, `gen_digits` returns a vector of ring elements $[c_0, \dots, c_{n-1}]$ such that $x = \sum_{i=0}^n c_i B^i$. Furthermore for all $i \neq n-1$, the elements c_i belonging to the fixed set of representatives of $R/(B)$.

8.5 Black box free \mathbf{Z}_p -modules.

(Very experimental)

```
GEN gen_ZpX_Dixon(GEN F, GEN V, GEN q, GEN p, long N, void *E, GEN lin(void *E, GEN
F, GEN z, GEN q), GEN invl(void *E, GEN z))
```

Let F be a `ZpXT` representing the coefficients of some abstract linear mapping f over $\mathbf{Z}_p[X]$ seen as a free \mathbf{Z}_p -module, let V be an element of $\mathbf{Z}_p[X]$ and let $q = p^N$. Return $y \in \mathbf{Z}_p[X]$ such that $f(y) = V \pmod{p^N}$ assuming the following holds for $n \leq N$:

- $\text{lin}(E, \text{FpX_red}(F, p^n), z, p^n) \equiv f(z) \pmod{p^n}$
- $f(\text{invl}(E, z)) \equiv z \pmod{p}$

The rationale for the argument F being that it allows `gen_ZpX_Dixon` to reduce it to the required p -adic precision.

```
GEN gen_ZpX_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q),
GEN invd(void *E, GEN b, GEN v, GEN q, long N))
```

Let x be an element of $\mathbf{Z}_p[X]$ seen as a free \mathbf{Z}_p -module, and f some differentiable function over $\mathbf{Z}_p[X]$ such that $f(x) \equiv 0 \pmod{p}$. Return y such that $f(y) \equiv 0 \pmod{p^n}$, assuming the following holds for all $a, b \in \mathbf{Z}_p[X]$ and $M \leq N$:

- $v = \text{eval}(E, a, p^N)$ is a vector of elements of $\mathbf{Z}_p[X]$,
- $w = \text{invd}(E, b, v, p^M, M)$ is an element in $\mathbf{Z}_p[X]$,
- $v[1] \equiv f(a) \pmod{p^N \mathbf{Z}_p[X]}$,
- $df_a(w) \equiv b \pmod{p^M \mathbf{Z}_p[X]}$

and df_a denotes the differential of f at a . Motivation: `eval` allows to evaluate f and `invd` allows to invert its differential. Frequently, data useful to compute the differential appear as a subproduct of computing the function. The vector v allows `eval` to provide these to `invd`. The implementation of `invd` will generally involves the use of the function `gen_ZpX_Dixon`.

Chapter 9: Operations on general PARI objects

9.1 Assignment.

It is in general easier to use a direct conversion, e.g. `y = stoi(s)`, than to allocate a target of correct type and sufficient size, then assign to it:

```
GEN y = cgeti(3); affsi(s, y);
```

These functions can still be moderately useful in complicated garbage collecting scenarios but you will be better off not using them.

`void gaffsg(long s, GEN x)` assigns the `long s` into the object `x`.

`void gaffect(GEN x, GEN y)` assigns the object `x` into the object `y`. Both `x` and `y` must be scalar types. Type conversions (e.g. from `t_INT` to `t_REAL` or `t_INTMOD`) occur if legitimate.

`int is_universal_constant(GEN x)` returns 1 if `x` is a global PARI constant you should never assign to (such as `gen_1`), and 0 otherwise.

9.2 Conversions.

9.2.1 Scalars.

`double rtodbl(GEN x)` applied to a `t_REAL x`, converts `x` into a `double` if possible.

`GEN dbltor(double x)` converts the `double x` into a `t_REAL`.

`long dblexpo(double x)` returns `expo(dbltor(x))`, but faster and without cluttering the stack.

`ulong dblmantissa(double x)` returns the most significant word in the mantissa of `dbltor(x)`.

`double gtodouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), converts `x` into a `double` if possible.

`long gtos(GEN x)` converts the `t_INT x` to a small integer if possible, otherwise raise an exception. This function is similar to `itos`, slightly slower since it checks the type of `x`.

`double dbllog2r(GEN x)` assuming that `x` is a non-zero `t_REAL`, returns an approximation to `log2(|x|)`.

`double dblmodulus(GEN x)` return an approximation to `|x|`.

`long gtolong(GEN x)` if `x` is an integer (not necessarily a `t_INT`), converts `x` into a `long` if possible.

`GEN fractor(GEN x, long l)` applied to a `t_FRAC x`, converts `x` into a `t_REAL` of length `prec`.

`GEN quadtofp(GEN x, long l)` applied to a `t_QUAD x`, converts `x` into a `t_REAL` or `t_COMPLEX` depending on the sign of the discriminant of `x`, to precision `l BITS_IN_LONG`-bit words.

GEN `upper_to_cx`(GEN `x`, long `*prec`) valid for a `t_COMPLEX` or `t_QUAD` belonging to the upper half-plane. If a `t_QUAD`, convert it to `t_COMPLEX` using accuracy `*prec`. If `x` is inexact, sets `*prec` to the precision of `x`.

GEN `cxtofp`(GEN `x`, long `prec`) converts the `t_COMPLEX` `x` to a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`).

GEN `cxcompotor`(GEN `x`, long `prec`) converts the `t_INT`, `t_REAL` or `t_FRAC` `x` to a `t_REAL` of length `prec`. These are all the real types which may occur as components of a `t_COMPLEX`; special case of `gtofp` (introduced so that the latter is not recursive and can thus be inlined).

GEN `cxtoreal`(GEN `x`) converts the complex (`t_INT`, `t_REAL`, `t_FRAC` or `t_COMPLEX`) `x` to a real number if its imaginary part is 0. Shallow function.

converts the `t_COMPLEX` `x` to a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`).

GEN `gtofp`(GEN `x`, long `prec`) converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of precision `prec`; not necessarily of *length* `prec`: a real 0 may be given as `real_0(...)`. If the result is a `t_COMPLEX` extra care is taken so that its modulus really has accuracy `prec`: there is a problem if the real part of the input is an exact 0; indeed, converting it to `real_0(prec)` would be wrong if the imaginary part is tiny, since the modulus would then become equal to 0, as in $1.E-100+0.E-28 = 0.E-28$.

GEN `gtomp`(GEN `z`, long `prec`) converts the real number `x` (`t_INT`, `t_REAL`, `t_FRAC`, real `t_QUAD`) to either a `t_INT` or a `t_REAL` of precision `prec`. Not memory clean if `x` is a `t_INT`: we return `x` itself and not a copy.

GEN `gcvtop`(GEN `x`, GEN `p`, long `l`) converts `x` into a `t_PADIC` of precision `l`. Works componentwise on recursive objects, e.g. `t_POL` or `t_VEC`. Converting 0 yields $O(p^l)$; converting a non-zero number yield a result well defined modulo $p^{v_p(x)+l}$.

GEN `cvtop`(GEN `x`, GEN `p`, long `l`) as `gcvtop`, assuming that `x` is a scalar.

GEN `cvtop2`(GEN `x`, GEN `y`) `y` being a p -adic, converts the scalar `x` to a p -adic of the same accuracy. Shallow function.

GEN `cvstop2`(long `s`, GEN `y`) `y` being a p -adic, converts the scalar `s` to a p -adic of the same accuracy. Shallow function.

GEN `gprec`(GEN `x`, long `l`) returns a copy of `x` whose precision is changed to `l` digits. The precision change is done recursively on all components of `x`. Digits means *decimal*, p -adic and X -adic digits for `t_REAL`, `t_SER`, `t_PADIC` components, respectively.

GEN `gprec_w`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision changed to `l words`. This is often more useful than `gprec`.

GEN `gprec_wtrunc`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision *truncated* to `l words`. Contrary to `gprec_w`, this function may never increase the precision of `x`.

GEN `gprec_wensure`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision *increased* to at least `l words`. Contrary to `gprec_w`, this function may never decrease the precision of `x`.

9.2.2 Modular objects / lifts.

GEN `gmodulo`(GEN `x`, GEN `y`) creates the object `Mod(x,y)` on the PARI stack, where `x` and `y` are either both `t_INTs`, and the result is a `t_INTMOD`, or `x` is a scalar or a `t_POL` and `y` a `t_POL`, and the result is a `t_POLMOD`.

GEN `gmodulgs`(GEN `x`, long `y`) same as `gmodulo` except `y` is a long.

GEN `gmodulsg`(long `x`, GEN `y`) same as `gmodulo` except `x` is a long.

GEN `gmodulss`(long `x`, long `y`) same as `gmodulo` except both `x` and `y` are longs.

GEN `lift_shallow`(GEN `x`) shallow version of `lift`

GEN `liftall_shallow`(GEN `x`) shallow version of `liftall`

GEN `liftint_shallow`(GEN `x`) shallow version of `liftint`

GEN `liftpol_shallow`(GEN `x`) shallow version of `liftpol`

GEN `centerlift0`(GEN `x`, long `v`) DEPRECATED, kept for backward compatibility only: use either `lift0(x,v)` or `centerlift(x)`.

9.2.3 Between polynomials and coefficient arrays.

GEN `gtopoly`(GEN `x`, long `v`) converts or truncates the object `x` into a `t_POL` with main variable number `v`. A common application would be the conversion of coefficient vectors (coefficients are given by decreasing degree). E.g. `[2,3]` goes to $2*v + 3$

GEN `gtopolyrev`(GEN `x`, long `v`) converts or truncates the object `x` into a `t_POL` with main variable number `v`, but vectors are converted in reverse order compared to `gtopoly` (coefficients are given by increasing degree). E.g. `[2,3]` goes to $3*v + 2$. In other words the vector represents a polynomial in the basis $(1, v, v^2, v^3, \dots)$.

GEN `normalizepol`(GEN `x`) applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. For internal use. Normalizing means deleting all leading *exact* zeroes (as per `isexactzero`), except if the polynomial turns out to be 0, in which case we try to find a coefficient `c` which is a non-rational zero, and return the constant polynomial `c`. (We do this so that information about the base ring is not lost.)

GEN `normalizepol_lg`(GEN `x`, long `l`) applies `normalizepol` to `x`, pretending that `lg(x)` is `l`, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `normalizepol(x)`.

GEN `normalizepol_approx`(GEN `x`, long `lx`) as `normalizepol_lg`, with the difference that we just delete all leading zeroes (as per `gequal0`). This rougher normalization is used when we have no other choice, for instance before attempting a Euclidean division by `x`.

The following routines do *not* copy coefficients on the stack (they only move pointers around), hence are very fast but not suitable for `gerepile` calls. Recall that an `RgV` (resp. an `RgX`, resp. an `RgM`) is a `t_VEC` or `t_COL` (resp. a `t_POL`, resp. a `t_MAT`) with arbitrary components. Similarly, an `RgXV` is a `t_VEC` or `t_COL` with `RgX` components, etc.

GEN `RgV_to_RgX`(GEN `x`, long `v`) converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopolyrev`, without copy).

GEN `RgV_to_RgX_reverse`(GEN `x`, long `v`) converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopoly`, without copy).

GEN `RgX_to_RgC`(GEN `x`, long `N`) converts the `t_POL` `x` to a `t_COL` `v` with `N` components. Coefficients of `x` are listed by increasing degree, so that `y[i]` is the coefficient of the term of degree $i - 1$ in `x`.

GEN `Rg_to_RgC`(GEN `x`, long `N`) as `RgX_to_RgV`, except that other types than `t_POL` are allowed for `x`, which is then considered as a constant polynomial.

GEN `RgM_to_RgXV`(GEN `x`, long `v`) converts the `RgM` `x` to a `t_VEC` of `RgX`, by repeated calls to `RgV_to_RgX`.

GEN `RgV_to_RgM`(GEN `v`, long `N`) converts the vector `v` to a `t_MAT` with `N` rows, by repeated calls to `Rg_to_RgV`.

GEN `RgXV_to_RgM`(GEN `v`, long `N`) converts the vector of `RgX` `v` to a `t_MAT` with `N` rows, by repeated calls to `RgX_to_RgV`.

GEN `RgM_to_RgXX`(GEN `x`, long `v`, long `w`) converts the `RgM` `x` into a `t_POL` in variable `v`, whose coefficients are `t_POL`s in variable `w`. This is a shortcut for

`RgV_to_RgX(RgM_to_RgXV(x, w), v);`

There are no consistency checks with respect to variable priorities: the above is an invalid object if `varncmp(v,w) ≥ 0`.

GEN `RgXX_to_RgM`(GEN `x`, long `N`) converts the `t_POL` `x` with `RgX` (or constant) coefficients to a matrix with `N` rows.

long `RgXY_degrees`(GEN `P`) return the degree of `P` with respect to the secondary variable.

GEN `RgXY_swap`(GEN `P`, long `n`, long `w`) converts the bivariate polynomial $P(u, v)$ (a `t_POL` with `t_POL` or scalar coefficients) to $P(\text{pol}_x[w], u)$, assuming `n` is an upper bound for $\deg_v(P)$.

GEN `RgXY_swapspec`(GEN `C`, long `n`, long `w`, long `lP`) as `RgXY_swap` where the coefficients of `P` are given by `gel(C,0), ..., gel(C,lP-1)`.

GEN `RgX_to_ser`(GEN `x`, long `l`) convert the `t_POL` `x` to a *shallow* `t_SER` of length $l \geq 2$. Unless the polynomial is an exact zero, the coefficient of lowest degree T^d of the result is not an exact zero (as per `isexactzero`). The remainder is $O(T^{d+l-2})$.

GEN `RgX_to_ser_inexact`(GEN `x`, long `l`) convert the `t_POL` `x` to a *shallow* `t_SER` of length $l \geq 2$. Unless the polynomial is zero, the coefficient of lowest degree T^d of the result is not zero (as per `equal0`). The remainder is $O(T^{d+l-2})$.

GEN `RgV_to_ser`(GEN `x`, long `v`, long `l`) convert the `t_VEC` `x`, to a *shallow* `t_SER` of length $l \geq 2$.

GEN `rfrac_to_ser`(GEN `F`, long `l`) applied to a `t_RFRAC` `F`, creates a `t_SER` of length $l \geq 2$ congruent to `F`. Not memory-clean but suitable for `gerepileupto`.

GEN `rfracrecip_to_ser_absolute`(GEN `F`, long `d`) applied to a `t_RFRAC` `F`, creates the `t_SER` $F(1/t) + O(t^d)$. Note that we use absolute and not relative precision here.

GEN `gtoser`(GEN `s`, long `v`, long `d`). This function is deprecated, kept for backward compatibility: it follows the semantic of `Ser(s,v)`, with $d = \text{seriesprecision}$ implied and is hard to use as a general conversion function. Use `gtoser_prec` instead.

It converts the object `s` into a `t_SER` with main variable number `v` and $d > 0$ significant terms, but the argument `d` is sometimes ignored. More precisely

- if s is a scalar (with respect to variable v), we return a constant power series with d significant terms;
- if s is a `t_POL` in variable v , it is truncated to d terms if needed;
- if s is a vector, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`), and the precision d is *ignored*;
- if s is already a power series in v , we return a copy, and the precision d is again *ignored*.

GEN `gtoser_prec`(GEN s , long v , long d) this function is a variant of `gtoser` following the semantic of `Ser(s,v,d)`: the precision d is always taken into account.

GEN `gtocol`(GEN x) converts the object x into a `t_COL`

GEN `gtomat`(GEN x) converts the object x into a `t_MAT`.

GEN `gtovec`(GEN x) converts the object x into a `t_VEC`.

GEN `gtovecsmall`(GEN x) converts the object x into a `t_VECSMALL`.

GEN `normalize`(GEN x) applied to an unnormalized `t_SER` x (i.e. type `t_SER` with all coefficients correctly set except that $x[2]$ might be zero), normalizes x correctly in place. Returns x . For internal use.

GEN `serchop0`(GEN s) given a `t_SER` of the form $x^v s(x)$, with $s(0) \neq 0$, return $x^v(s-s(0))$. Shallow function.

GEN `serchop_i`(GEN x , long n) returns a shallow copy of `t_SER` x with all terms of degree strictly less than n removed. Shallow version of `serchop`.

9.3 Constructors.

9.3.1 Clean constructors.

GEN `zeropadic`(GEN p , long n) creates a 0 `t_PADIC` equal to $O(\mathfrak{p}^n)$.

GEN `zeroser`(long v , long n) creates a 0 `t_SER` in variable v equal to $O(X^n)$.

GEN `scalarser`(GEN x , long v , long $prec$) creates a constant `t_SER` in variable v and precision $prec$, whose constant coefficient is (a copy of) x , in other words $x + O(v^{prec})$. Assumes that $prec \geq 0$.

GEN `pol_0`(long v) Returns the constant polynomial 0 in variable v .

GEN `pol_1`(long v) Returns the constant polynomial 1 in variable v .

GEN `pol_x`(long v) Returns the monomial of degree 1 in variable v .

GEN `pol_xn`(long n , long v) Returns the monomial of degree n in variable v ; assume that $n \geq 0$.

GEN `pol_xnall`(long n , long v) Returns the Laurent monomial of degree n in variable v ; $n < 0$ is allowed.

GEN `pol_x_powers`(long N , long v) returns the powers of `pol_x(v)`, of degree 0 to $N - 1$, in a vector with N components.

GEN `scalarspol`(GEN x , long v) creates a constant `t_POL` in variable v , whose constant coefficient is (a copy of) x .

GEN `deg1pol(GEN a, GEN b, long v)` creates the degree 1 `t_POL` $ap_1x(v) + b$
 GEN `zeropol(long v)` is identical `pol_0`.
 GEN `zerocol(long n)` creates a `t_COL` with `n` components set to `gen_0`.
 GEN `zerovec(long n)` creates a `t_VEC` with `n` components set to `gen_0`.
 GEN `zerovec_block(long n)` as `zerovec` but return a clone.
 GEN `col_ei(long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).
 GEN `vec_ei(long n, long i)` creates a `t_VEC` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).
 GEN `trivial_fact(void)` returns the trivial (empty) factorization `Mat([]~, []~)`
 GEN `prime_fact(GEN x)` returns the factorization `Mat([x]~, [1]~)`
 GEN `Rg_col_ei(GEN x, long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `x`.
 GEN `vecsmall_ei(long n, long i)` creates a `t_VECSMALL` with `n` components set to 0, but for the `i`-th one which is set to 1 (`i`-th vector in the canonical basis).
 GEN `scalarcol(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a copy of `x`. (The name comes from `RgV_isscalar`.)
 GEN `mkintmodu(ulong x, ulong y)` creates the `t_INTMOD` `Mod(x, y)`. The inputs must satisfy $x < y$.
 GEN `zeromat(long m, long n)` creates a `t_MAT` with `m` x `n` components set to `gen_0`. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns. To fully allocate a matrix initialized with zero entries, use `zeromatcopy`.
 GEN `zeromatcopy(long m, long n)` creates a `t_MAT` with `m` x `n` components set to `gen_0`.
 GEN `matid(long n)` identity matrix in dimension `n` (with components `gen_1` and `gen_0`).
 GEN `scalarmat(GEN x, long n)` scalar matrix, `x` times the identity.
 GEN `scalarmat_s(long x, long n)` scalar matrix, `stoi(x)` times the identity.
 GEN `vecrange(GEN a, GEN b)` returns the `t_VEC` `[a..b]`.
 GEN `vecrangess(long a, long b)` returns the `t_VEC` `[a..b]`.

See also next section for analogs of the following functions:

GEN `mkfracss(long x, long y)` creates the `t_FRAC` x/y . Assumes that $y > 1$ and $(x, y) = 1$.
 GEN `sstoQ(long x, long y)` returns the `t_INT` or `t_FRAC` x/y ; no assumptions.
 void `Qtoss(GEN q, long *n, long *d)` given a `t_INT` or `t_FRAC` `q`, set `n` and `d` such that $q = n/d$ with $d \geq 1$ and $(n, d) = 1$. Overflow error if numerator or denominator do not fit into a long integer.
 GEN `mkfraccopy(GEN x, GEN y)` creates the `t_FRAC` x/y . Assumes that $y > 1$ and $(x, y) = 1$.
 GEN `mkrfraccopy(GEN x, GEN y)` creates the `t_RFRAC` x/y . Assumes that `y` is a `t_POL`, `x` a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

GEN `mkcolcopy(GEN x)` creates a 1-dimensional `t_COL` containing `x`.

GEN `mkmatcopy(GEN x)` creates a 1-by-1 `t_MAT` wrapping the `t_COL` `x`.

GEN `mkveccopy(GEN x)` creates a 1-dimensional `t_VEC` containing `x`.

GEN `mkvec2copy(GEN x, GEN y)` creates a 2-dimensional `t_VEC` equal to `[x,y]`.

GEN `mkcols(long x)` creates a 1-dimensional `t_COL` containing `stoi(x)`.

GEN `mkcol2s(long x, long y)` creates a 2-dimensional `t_COL` containing `[stoi(x), stoi(y)]` .

GEN `mkcol3s(long x, long y, long z)` creates a 3-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z)]` .

GEN `mkcol4s(long x, long y, long z, long t)` creates a 4-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]` .

GEN `mkvecs(long x)` creates a 1-dimensional `t_VEC` containing `stoi(x)`.

GEN `mkvec2s(long x, long y)` creates a 2-dimensional `t_VEC` containing `[stoi(x), stoi(y)]`.

GEN `mkmat22s(long a, long b, long c, long d)` creates the 2 by 2 `t_MAT` with successive rows `[stoi(a), stoi(b)]` and `[stoi(c), stoi(d)]`.

GEN `mkvec3s(long x, long y, long z)` creates a 3-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z)]`.

GEN `mkvec4s(long x, long y, long z, long t)` creates a 4-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]`.

GEN `mkvecsmall(long x)` creates a 1-dimensional `t_VECSMALL` containing `x`.

GEN `mkvecsmall2(long x, long y)` creates a 2-dimensional `t_VECSMALL` containing `[x, y]`.

GEN `mkvecsmall3(long x, long y, long z)` creates a 3-dimensional `t_VECSMALL` containing `[x, y, z]`.

GEN `mkvecsmall4(long x, long y, long z, long t)` creates a 4-dimensional `t_VECSMALL` containing `[x, y, z, t]`.

GEN `mkvecsmalln(long n, ...)` returns the `t_VECSMALL` whose `n` coefficients (`long`) follow.
Warning: since this is a variadic function, C type promotion is not performed on the arguments by the compiler, thus you have to make sure that all the arguments are of type `long`, in particular integer constants need to be written with the L suffix: `mkvecsmalln(2, 1L, 2L)` is correct, but `mkvecsmalln(2, 1, 2)` is not.

9.3.2 Unclean constructors.

Contrary to the policy of general PARI functions, the functions in this subsection do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. In particular, they are faster than their clean equivalent (which may not exist). *If* you restrict their arguments to universal objects (e.g `gen_0`), then the above warning does not apply.

`GEN mkcomplex(GEN x, GEN y)` creates the `t_COMPLEX` $x + iy$.

`GEN mulcxI(GEN x)` creates the `t_COMPLEX` ix . The result in general contains data pointing back to the original x . Use `gcopy` if this is a problem. But in most cases, the result is to be used immediately, before x is subject to garbage collection.

`GEN mulcxmI(GEN x)`, as `mulcxI`, but returns $-ix$.

`GEN mulcxpowIs(GEN x, long k)`, as `mulcxI`, but returns $x \cdot i^k$.

`GEN mkquad(GEN n, GEN x, GEN y)` creates the `t_QUAD` $x + yw$, where w is a root of n , which is of the form `quadpoly(D)`.

`GEN mkfrac(GEN x, GEN y)` creates the `t_FRAC` x/y . Assumes that $y > 1$ and $(x, y) = 1$.

`GEN mkrfrac(GEN x, GEN y)` creates the `t_RFRAC` x/y . Assumes that y is a `t_POL`, x a compatible type whose variable has lower or same priority, with $(x, y) = 1$.

`GEN mkcol(GEN x)` creates a 1-dimensional `t_COL` containing x .

`GEN mkcol2(GEN x, GEN y)` creates a 2-dimensional `t_COL` equal to $[x, y]$.

`GEN mkcol3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_COL` equal to $[x, y, z]$.

`GEN mkcol4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_COL` equal to $[x, y, z, t]$.

`GEN mkcol5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_COL` equal to $[a_1, a_2, a_3, a_4, a_5]$.

`GEN mkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` creates the 6-dimensional column vector $[x, y, z, t, u, v]$.

`GEN mkintmod(GEN x, GEN y)` creates the `t_INTMOD` $\text{Mod}(x, y)$. The inputs must be `t_INTs` satisfying $0 \leq x < y$.

`GEN mkpolmod(GEN x, GEN y)` creates the `t_POLMOD` $\text{Mod}(x, y)$. The input must satisfy $\deg x < \deg y$ with respect to the main variable of the `t_POL` y . x may be a scalar.

`GEN mkmat(GEN x)` creates a 1-column `t_MAT` with column x (a `t_COL`).

`GEN mkmat2(GEN x, GEN y)` creates a 2-column `t_MAT` with columns x, y (`t_COLS` of the same length).

`GEN mkmat22(GEN a, GEN b, GEN c, GEN d)` creates the 2 by 2 `t_MAT` with successive rows $[a, b]$ and $[c, d]$.

`GEN mkmat3(GEN x, GEN y, GEN z)` creates a 3-column `t_MAT` with columns x, y, z (`t_COLS` of the same length).

`GEN mkmat4(GEN x, GEN y, GEN z, GEN t)` creates a 4-column `t_MAT` with columns x, y, z, t (`t_COLS` of the same length).

`GEN mkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` creates a 5-column `t_MAT` with columns x, y, z, t, u (`t_COLS` of the same length).

GEN `mkvec`(GEN `x`) creates a 1-dimensional `t_VEC` containing `x`.

GEN `mkvec2`(GEN `x`, GEN `y`) creates a 2-dimensional `t_VEC` equal to `[x,y]`.

GEN `mkvec3`(GEN `x`, GEN `y`, GEN `z`) creates a 3-dimensional `t_VEC` equal to `[x,y,z]`.

GEN `mkvec4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) creates a 4-dimensional `t_VEC` equal to `[x,y,z,t]`.

GEN `mkvec5`(GEN `a1`, GEN `a2`, GEN `a3`, GEN `a4`, GEN `a5`) creates the 5-dimensional `t_VEC` equal to `[a1,a2,a3,a4,a5]`.

GEN `mkqfi`(GEN `x`, GEN `y`, GEN `z`) creates `t_QFI` equal to `Qfb(x,y,z)`, assuming that $y^2 - 4xz < 0$.

GEN `mkerr`(long `n`) returns a `t_ERROR` with error code `n` (enum `err_list`).

It is sometimes useful to return such a container whose entries are not universal objects, but nonetheless suitable for `gerepileupto`. If the entries can be computed at the time the result is returned, the following macros achieve this effect:

GEN `retmkvec`(GEN `x`) returns a vector containing the single entry `x`, where the vector root is created just before the function argument `x` is evaluated. Expands to

```
{
  GEN res = cgetg(2, t_VEC);
  gel(res, 1) = x; /* or rather, the expansion of x */
  return res;
}
```

For instance, the `retmkvec(gcopy(x))` returns a clean object, just like `return mkveccopy(x)` would.

GEN `retmkvec2`(GEN `x`, GEN `y`) returns the 2-dimensional `t_VEC` `[x,y]`.

GEN `retmkvec3`(GEN `x`, GEN `y`, GEN `z`) returns the 3-dimensional `t_VEC` `[x,y,z]`.

GEN `retmkvec4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) returns the 4-dimensional `t_VEC` `[x,y,z,t]`.

GEN `retmkvec5`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`) returns the 5-dimensional row vector `[x,y,z,t,u]`.

GEN `retconst_vec`(long `n`, GEN `x`) returns the `n`-dimensional `t_VEC` whose entries are constant and all equal to `x`.

GEN `retmkcol`(GEN `x`) returns the 1-dimensional `t_COL` `[x]` .

GEN `retmkcol2`(GEN `x`, GEN `y`) returns the 2-dimensional `t_COL` `[x,y]` .

GEN `retmkcol3`(GEN `x`, GEN `y`, GEN `z`) returns the 3-dimensional `t_COL` `[x,y,z]` .

GEN `retmkcol4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) returns the 4-dimensional `t_COL` `[x,y,z,t]` .

GEN `retmkcol5`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`) returns the 5-dimensional column vector `[x,y,z,t,u]` .

GEN `retmkcol6`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`, GEN `v`) returns the 6-dimensional column vector `[x,y,z,t,u,v]` .

GEN `retconst_col`(long `n`, GEN `x`) returns the `n`-dimensional `t_COL` whose entries are constant and all equal to `x`.

GEN `retmkmat`(GEN `x`) returns the 1-column `t_MAT` with column `x`.

GEN `retmkmat2`(GEN `x`, GEN `y`) returns the 2-column `t_MAT` with columns `x`, `y`.

GEN `retmkmat3`(GEN `x`, GEN `y`, GEN `z`) returns the 3-dimensional `t_MAT` with columns `x`, `y`, `z`.

GEN `retmkmat4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) returns the 4-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`.

GEN `retmkmat5`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`) returns the 5-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`, `u`.

GEN `retmkcomplex`(GEN `x`, GEN `y`) returns the `t_COMPLEX` $x + I*y$.

GEN `retmkfrac`(GEN `x`, GEN `y`) returns the `t_FRAC` x / y . Assume x and y are coprime and $y > 1$.

GEN `retmkfrac`(GEN `x`, GEN `y`) returns the `t_RFRAC` x / y . Assume x and y are coprime and more generally that the rational function cannot be simplified.

GEN `retmkintmod`(GEN `x`, GEN `y`) returns the `t_INTMOD` `Mod(x, y)`.

GEN `retmkqfi`(GEN `a`, GEN `b`, GEN `c`).

GEN `retmkqfr`(GEN `a`, GEN `b`, GEN `c`, GEN `d`).

GEN `retmkquad`(GEN `n`, GEN `a`, GEN `b`).

GEN `retmkpolmod`(GEN `x`, GEN `y`) returns the `t_POLMOD` `Mod(x, y)`.

GEN `mkintn`(long `n`, ...) returns the non-negative `t_INT` whose development in base 2^{32} is given by the following n 32bit-words (unsigned int).

```
mkintn(3, a2, a1, a0);
```

returns $a_2 2^{64} + a_1 2^{32} + a_0$.

GEN `mkpoln`(long `n`, ...) Returns the `t_POL` whose n coefficients (GEN) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial $X^2 + 2X$ (in variable 0, use `setvarn` if you want other variable numbers). Beware that n is the number of coefficients, hence *one more* than the degree.

GEN `mkvecn`(long `n`, ...) returns the `t_VEC` whose n coefficients (GEN) follow.

GEN `mkcoln`(long `n`, ...) returns the `t_COL` whose n coefficients (GEN) follow.

GEN `scalarcoll_shallow`(GEN `x`, long `n`) creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a shallow copy of `x`. (The name comes from `RgV_isscalar`.)

GEN `scalarmat_shallow`(GEN `x`, long `n`) creates an $n \times n$ scalar matrix whose diagonal is set to shallow copies of the scalar `x`.

GEN `RgX_sylvestermatrix`(GEN `f`, GEN `g`) return the Sylvester matrix attached to the two `t_POL` in the same variable f and g .

GEN `diagonal_shallow`(GEN `x`) returns a diagonal matrix whose diagonal is given by the vector x . Shallow function.

GEN `scalarpoll_shallow`(GEN `a`, long `v`) returns the degree 0 `t_POL` `apol_x(v)`⁰.

GEN `deg1pol_shallow`(GEN `a`, GEN `b`, long `v`) returns the degree 1 `t_POL` $a\text{pol_x}(v) + b$

GEN `deg2pol_shallow`(GEN `a`, GEN `b`, GEN `c`, long `v`) returns the degree 2 `t_POL` $ax^2 + bx + c$ where $x = \text{pol_x}(v)$.

GEN `zeropadic_shallow`(GEN `p`, long `n`) returns a (shallow) 0 `t_PADIC` equal to $O(p^n)$.

9.3.3 From roots to polynomials.

GEN `deg1_from_roots`(GEN `L`, long `v`) given a vector L of scalars, returns the vector of monic linear polynomials in variable v whose roots are the $L[i]$, i.e. the $x - L[i]$.

GEN `roots_from_deg1`(GEN `L`) given a vector L of monic linear polynomials, return their roots, i.e. the $-L[i](0)$.

GEN `roots_to_pol`(GEN `L`, long `v`) given a vector of scalars L , returns the monic polynomial in variable v whose roots are the $L[i]$. Leaves some garbage on stack, but suitable for `gerepileupto`.

GEN `roots_to_pol_r1`(GEN `L`, long `v`, long `r1`) as `roots_to_pol` assuming the first r_1 roots are “real”, and the following ones are representatives of conjugate pairs of “complex” roots. So if L has $r_1 + r_2$ elements, we obtain a polynomial of degree $r_1 + 2r_2$. In most applications, the roots are indeed real and complex, but the implementation assumes only that each “complex” root z introduces a quadratic factor $X^2 - \text{trace}(z)X + \text{norm}(z)$. Leaves some garbage on stack, but suitable for `gerepileupto`.

9.4 Integer parts.

GEN `gffloor`(GEN `x`) creates the floor of x , i.e. the (true) integral part.

GEN `gfrac`(GEN `x`) creates the fractional part of x , i.e. x minus the floor of x .

GEN `gceil`(GEN `x`) creates the ceiling of x .

GEN `ground`(GEN `x`) rounds towards $+\infty$ the components of x to the nearest integers.

GEN `grndtoi`(GEN `x`, long `*e`) same as `ground`, but in addition sets `*e` to the binary exponent of $x - \text{ground}(x)$. If this is positive, all significant bits are lost. This kind of situation raises an error message in `ground` but not in `grndtoi`.

GEN `gtrunc`(GEN `x`) truncates x . This is the false integer part if x is a real number (i.e. the unique integer closest to x among those between 0 and x). If x is a `t_SER`, it is truncated to a `t_POL`; if x is a `t_RFRAC`, this takes the polynomial part.

GEN `gtrunc2n`(GEN `x`, long `n`) creates the floor of $2^n x$, this is only implemented for `t_INT`, `t_REAL`, `t_FRAC` and `t_COMPLEX` of those.

GEN `gcvtoi`(GEN `x`, long `*e`) analogous to `grndtoi` for `t_REAL` inputs except that rounding is replaced by truncation. Also applies componentwise for vector or matrix inputs; otherwise, sets `*e` to `-HIGHEXPOBIT` (infinite real accuracy) and return `gtrunc(x)`.

9.5 Valuation and shift.

GEN `gshift[z](GEN x, long n[, GEN z]`) yields the result of shifting (the components of) `x` left by `n` (if `n` is non-negative) or right by $-n$ (if `n` is negative). Applies only to `t_INT` and vectors/matrices of such. For other types, it is simply multiplication by 2^n .

GEN `gmul2n[z](GEN x, long n[, GEN z]`) yields the product of `x` and 2^n . This is different from `gshift` when `n` is negative and `x` is a `t_INT`: **gshift** truncates, while **gmul2n** creates a fraction if necessary.

long `gvaluation(GEN x, GEN p)` returns the greatest exponent e such that p^e divides `x`, when this makes sense.

long `gval(GEN x, long v)` returns the highest power of the variable number `v` dividing the `t_POL` `x`.

9.6 Comparison operators.

9.6.1 Generic.

long `gcmp(GEN x, GEN y)` comparison of `x` with `y`: returns 1 ($x > y$), 0 ($x = y$) or -1 ($x < y$). Two `t_STR` are compared using the standard lexicographic ordering; a `t_STR` cannot be compared to any non-string type. If neither `x` nor `y` is a `t_STR`, their allowed types are `t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` with positive discriminant (use the canonical embedding $w \rightarrow \sqrt{D}/2$ or $w \rightarrow (1 + \sqrt{D})/2$) or `t_INFINITY`. Use `cmp_universal` to compare arbitrary GENs.

long `lexcmp(GEN x, GEN y)` comparison of `x` with `y` for the lexicographic ordering; when comparing objects of different lengths whose components are all equal up to the smallest of their length, consider that the longest is largest. Consider scalars as 1-component vectors. Return `gcmp(x, y)` if both arguments are scalars.

int `gequalX(GEN x)` return 1 (true) if `x` is a variable (monomial of degree 1 with `t_INT` coefficients equal to 1 and 0), and 0 otherwise

long `gequal(GEN x, GEN y)` returns 1 (true) if `x` is equal to `y`, 0 otherwise. A priori, this makes sense only if `x` and `y` have the same type, in which case they are recursively compared componentwise. When the types are different, a **true** result means that `x - y` was successfully computed and that `gequal0` found it equal to 0. In particular

`gequal(cgetg(1, t_VEC), gen_0)`

is true, and the relation is not transitive. E.g. an empty `t_COL` and an empty `t_VEC` are not equal but are both equal to `gen_0`.

long `gidentical(GEN x, GEN y)` returns 1 (true) if `x` is identical to `y`, 0 otherwise. In particular, the types and length of `x` and `y` must be equal. This test is much stricter than `gequal`, in particular, `t_REAL` with different accuracies are tested different. This relation is transitive.

GEN `gmax(GEN x, GEN y)` returns a copy of the maximum of x and y , compared using `gcmp`.

GEN `gmin(GEN x, GEN y)` returns a copy of the minimum of x and y , compared using `gcmp`.

GEN `gmax_shallow(GEN x, GEN y)` shallow version of `gmax`.

GEN `gmin_shallow(GEN x, GEN y)` shallow version of `gmin`.

9.6.2 Comparison with a small integer.

`int isexactzero(GEN x)` returns 1 (true) if `x` is exactly equal to 0 (including `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. This includes recursive objects, for instance vectors, whose components are 0.

`GEN gisexactzero(GEN x)` returns `NULL` unless `x` is exactly equal to 0 (as per `isexactzero`). When `x` is an exact zero return the attached scalar zero as a `t_INT` (`gen_0`), a `t_INTMOD` (`Mod(0,N)` for the largest possible `N`) or a `t_FFELT`.

`int isrationalzero(GEN x)` returns 1 (true) if `x` is equal to an integer 0 (excluding `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. Contrary to `isintzero`, this includes recursive objects, for instance vectors, whose components are 0.

`int ismpzero(GEN x)` returns 1 (true) if `x` is a `t_INT` or a `t_REAL` equal to 0.

`int isintzero(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 0.

`int isint1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 1.

`int isintm1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to -1 .

`int equali1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to 1, and return 0 (false) otherwise.

`int equalim1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to -1 , and return 0 (false) otherwise.

`int is_pm1(GEN x)`. Assuming that `x` is a *non-zero* `t_INT`, return 1 (true) if `x` is equal to -1 or 1, and return 0 (false) otherwise.

`int gequal0(GEN x)` returns 1 (true) if `x` is equal to 0, 0 (false) otherwise.

`int gequal1(GEN x)` returns 1 (true) if `x` is equal to 1, 0 (false) otherwise.

`int gequalm1(GEN x)` returns 1 (true) if `x` is equal to -1 , 0 (false) otherwise.

`long gcmpsg(long s, GEN x)`

`long gcmpgs(GEN x, long s)` comparison of `x` with the `long s`.

`GEN gmaxsg(long s, GEN x)`

`GEN gmaxgs(GEN x, long s)` returns the largest of `x` and the `long s` (converted to `GEN`)

`GEN gminsg(long s, GEN x)`

`GEN gminggs(GEN x, long s)` returns the smallest of `x` and the `long s` (converted to `GEN`)

`long gequalsg(long s, GEN x)`

`long gequalgs(GEN x, long s)` returns 1 (true) if `x` is equal to the `long s`, 0 otherwise.

9.7 Miscellaneous Boolean functions.

`int isrationalzeroscalar(GEN x)` equivalent to, but faster than,

```
is_scalar_t(typ(x)) && isrationalzero(x)
```

`int isinexact(GEN x)` returns 1 (true) if x has an inexact component, and 0 (false) otherwise.

`int isinexactreal(GEN x)` return 1 if x has an inexact `t_REAL` component, and 0 otherwise.

`int isrealappr(GEN x, long e)` applies (recursively) to complex inputs; returns 1 if x is approximately real to the bit accuracy e , and 0 otherwise. This means that any `t_COMPLEX` component must have imaginary part t satisfying `gexpo(t) < e`.

`int isint(GEN x, GEN *n)` returns 0 (false) if x does not round to an integer. Otherwise, returns 1 (true) and set n to the rounded value.

`int issmall(GEN x, long *n)` returns 0 (false) if x does not round to a small integer (suitable for `itos`). Otherwise, returns 1 (true) and set n to the rounded value.

`long iscomplex(GEN x)` returns 1 (true) if x is a complex number (of component types embeddable into the reals) but is not itself real, 0 if x is a real (not necessarily of type `t_REAL`), or raises an error if x is not embeddable into the complex numbers.

9.7.1 Obsolete.

The following less convenient comparison functions and Boolean operators were used by the historical GP interpreter. They are provided for backward compatibility only and should not be used:

```
GEN gle(GEN x, GEN y)
```

```
GEN glt(GEN x, GEN y)
```

```
GEN gge(GEN x, GEN y)
```

```
GEN ggt(GEN x, GEN y)
```

```
GEN geq(GEN x, GEN y)
```

```
GEN gne(GEN x, GEN y)
```

```
GEN gor(GEN x, GEN y)
```

```
GEN gand(GEN x, GEN y)
```

```
GEN gnot(GEN x, GEN y)
```

9.8 Sorting.

9.8.1 Basic sort.

GEN `sort(GEN x)` sorts the vector x in ascending order using a mergesort algorithm, and `gcmp` as the underlying comparison routine (returns the sorted vector). This routine copies all components of x , use `gen_sort_inplace` for a more memory-efficient function.

GEN `lexsort(GEN x)`, as `sort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

GEN `vecsrt(GEN x, GEN k)`, as `sort`, but sorts the vector x in ascending *lexicographic* order, according to the entries of the `t_VECSMALL` k . For example, if $k = [2, 1, 3]$, sorting will be done with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

9.8.2 Indirect sorting.

GEN `indexsort(GEN x)` as `sort`, but only returns the permutation which, applied to x , would sort the vector. The result is a `t_VECSMALL`.

GEN `indexlexsort(GEN x)`, as `indexsort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

GEN `indexvecsrt(GEN x, GEN k)`, as `vecsrt`, but only returns the permutation that would sort the vector x .

long `vecindexmin(GEN x)` returns the index for a maximal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

long `vecindexmax(GEN x)` returns the index for a maximal element of x (`t_VEC`, `t_COL` or `t_VECSMALL`).

long `vecindexmax(GEN x)`

9.8.3 Generic sort and search. The following routines allow to use an arbitrary comparison function `int (*cmp)(void* data, GEN x, GEN y)`, such that `cmp(data,x,y)` returns a negative result if $x < y$, a positive one if $x > y$ and 0 if $x = y$. The `data` argument is there in case your `cmp` requires additional context.

GEN `gen_sort(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `sort`, with an explicit comparison routine.

GEN `gen_sort_uniq(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `gen_sort`, removing duplicate entries.

GEN `gen_indexsort(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`.

GEN `gen_indexsort_uniq(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`, removing duplicate entries.

void `gen_sort_inplace(GEN x, void *data, int (*cmp)(void*, GEN, GEN), GEN *perm)` sort x in place, without copying its components. If `perm` is non-NULL, it is set to the permutation that would sort the original x .

GEN `gen_setminus(GEN A, GEN B, int (*cmp)(GEN, GEN))` given two sorted vectors A and B , returns the vector of elements of A not belonging to B .

`GEN sort_factor(GEN y, void *data, int (*cmp)(void *, GEN, GEN))`: assuming `y` is a factorization matrix, sorts its rows in place (no copy is made) according to the comparison function `cmp` applied to its first column.

`GEN merge_sort_uniq(GEN x, GEN y, void *data, int (*cmp)(void *, GEN, GEN))` assuming `x` and `y` are sorted vectors, with respect to the `cmp` comparison function, return a sorted concatenation, with duplicates removed.

`GEN merge_factor(GEN fx, GEN fy, void *data, int (*cmp)(void *, GEN, GEN))` let `fx` and `fy` be factorization matrices for X and Y sorted with respect to the comparison function `cmp` (see `sort_factor`), returns the factorization of $X * Y$.

`long gen_search(GEN v, GEN y, long flag, void *data, int (*cmp)(void*, GEN, GEN))`
.

Let `v` be a vector sorted according to `cmp(data,a,b)`; look for an index i such that `v[i]` is equal to `y`. `flag` has the same meaning as in `setsearch`: if `flag` is 0, return i if it exists and 0 otherwise; if `flag` is non-zero, return 0 if i exists and the index where `y` should be inserted otherwise.

`long tablesearch(GEN T, GEN x, int (*cmp)(GEN, GEN))` is a faster implementation for the common case `gen_search(T,x,0,cmp,cmp_nodata)`.

9.8.4 Further useful comparison functions.

`int cmp_universal(GEN x, GEN y)` a somewhat arbitrary universal comparison function, devoid of sensible mathematical meaning. It is transitive, and returns 0 if and only if `gidentical(x,y)` is true. Useful to sort and search vectors of arbitrary data.

`int cmp_nodata(void *data, GEN x, GEN y)`. This function is a hack used to pass an existing basic comparison function lacking the `data` argument, i.e. with prototype `int (*cmp)(GEN x, GEN y)`. Instead of `gen_sort(x, NULL, cmp)` which may or may not work depending on how your compiler handles typecasts between incompatible function pointers, one should use `gen_sort(x, (void*)cmp, cmp_nodata)`.

Here are a few basic comparison functions, to be used with `cmp_nodata`:

`int ZV_cmp(GEN x, GEN y)` compare two `ZV`, which we assume have the same length (lexicographic order).

`int cmp_Flx(GEN x, GEN y)` compare two `Flx`, which we assume have the same main variable (lexicographic order).

`int cmp_RgX(GEN x, GEN y)` compare two polynomials, which we assume have the same main variable (lexicographic order). The coefficients are compared using `gcmp`.

`int cmp_prime_over_p(GEN x, GEN y)` compare two prime ideals, which we assume divide the same prime number. The comparison is ad hoc but orders according to increasing residue degrees.

`int cmp_prime_ideal(GEN x, GEN y)` compare two prime ideals in the same nf . Orders by increasing primes, breaking ties using `cmp_prime_over_p`.

`int cmp_padic(GEN x, GEN y)` compare two `t_PADIC` (for the same prime p).

Finally a more elaborate comparison function:

`int gen_cmp_RgX(void *data, GEN x, GEN y)` compare two polynomials, ordering first by increasing degree, then according to the coefficient comparison function:

```
int (*cmp_coeff)(GEN,GEN) = (int (*)(GEN,GEN)) data;
```

9.9 Divisibility, Euclidean division.

GEN `gdivexact`(GEN `x`, GEN `y`) returns the quotient x/y , assuming `y` divides `x`. Not stack clean if $y = 1$ (we return `x`, not a copy).

int `gdvd`(GEN `x`, GEN `y`) returns 1 (true) if `y` divides `x`, 0 otherwise.

GEN `gdiventres`(GEN `x`, GEN `y`) creates a 2-component vertical vector whose components are the true Euclidean quotient and remainder of `x` and `y`.

GEN `gdivent`[`z`](GEN `x`, GEN `y`[, GEN `z`]) yields the true Euclidean quotient of `x` and the `t_INT` or `t_POL` `y`, as per the `\ GP` operator.

GEN `gdiventsg`(long `s`, GEN `y`[, GEN `z`]), as `gdivent` except that `x` is a long.

GEN `gdiventgs`[`z`](GEN `x`, long `s`[, GEN `z`]), as `gdivent` except that `y` is a long.

GEN `gmod`[`z`](GEN `x`, GEN `y`[, GEN `z`]) yields the remainder of `x` modulo the `t_INT` or `t_POL` `y`, as per the `% GP` operator. A `t_REAL` or `t_FRAC` `y` is also allowed, in which case the remainder is the unique real r such that $0 \leq r < |y|$ and $y = qx + r$ for some (in fact unique) integer q .

GEN `gmodsg`(long `s`, GEN `y`[, GEN `z`]) as `gmod`, except `x` is a long.

GEN `gmodgs`(GEN `x`, long `s`[, GEN `z`]) as `gmod`, except `y` is a long.

GEN `gdivmod`(GEN `x`, GEN `y`, GEN `*r`) If `r` is not equal to `NULL` or `ONLY_REM`, creates the (false) Euclidean quotient of `x` and `y`, and puts (the address of) the remainder into `*r`. If `r` is equal to `NULL`, do not create the remainder, and if `r` is equal to `ONLY_REM`, create and output only the remainder. The remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`.

GEN `poldivrem`(GEN `x`, GEN `y`, GEN `*r`) same as `gdivmod` but specifically for `t_POLs` `x` and `y`, not necessarily in the same variable. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

GEN `gdeuc`(GEN `x`, GEN `y`) creates the Euclidean quotient of the `t_POLs` `x` and `y`. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

GEN `grem`(GEN `x`, GEN `y`) creates the Euclidean remainder of the `t_POL` `x` divided by the `t_POL` `y`. Either of `x` and `y` may also be scalars, treated as polynomials of degree 0.

GEN `gdivround`(GEN `x`, GEN `y`) if `x` and `y` are real (`t_INT`, `t_REAL`, `t_FRAC`), return the rounded Euclidean quotient of `x` and `y` as per the `\ / GP` operator. Operate componentwise if `x` is a `t_COL`, `t_VEC` or `t_MAT`. Otherwise as `gdivent`.

GEN `centermod_i`(GEN `x`, GEN `y`, GEN `y2`), as `centermodii`, componentwise.

GEN `centermod`(GEN `x`, GEN `y`), as `centermod_i`, except that `y2` is computed (and left on the stack for efficiency).

GEN `ginvmod`(GEN `x`, GEN `y`) creates the inverse of `x` modulo `y` when it exists. `y` must be of type `t_INT` (in which case `x` is of type `t_INT`) or `t_POL` (in which case `x` is either a scalar type or a `t_POL`).

9.10 GCD, content and primitive part.

9.10.1 Generic.

GEN `resultant`(GEN `x`, GEN `y`) creates the resultant of the `t_POLs` `x` and `y` computed using Sylvester's matrix (inexact inputs), a modular algorithm (inputs in $\mathbf{Q}[X]$) or the subresultant algorithm, as optimized by Lazard and Ducos. Either of `x` and `y` may also be scalars (treated as polynomials of degree 0)

GEN `ggcd`(GEN `x`, GEN `y`) creates the GCD of `x` and `y`.

GEN `glcm`(GEN `x`, GEN `y`) creates the LCM of `x` and `y`.

GEN `gbezout`(GEN `x`, GEN `y`, GEN `*u`, GEN `*v`) returns the GCD of `x` and `y`, and puts (the addresses of) objects `u` and `v` such that $ux + vy = \text{gcd}(x, y)$ into `*u` and `*v`.

GEN `subresext`(GEN `x`, GEN `y`, GEN `*U`, GEN `*V`) returns the resultant of `x` and `y`, and puts (the addresses of) polynomials `u` and `v` such that $ux + vy = \text{Res}(x, y)$ into `*U` and `*V`.

GEN `content`(GEN `x`) returns the GCD of all the components of `x`.

GEN `primitive_part`(GEN `x`, GEN `*c`) sets `c` to `content(x)` and returns the primitive part x / c . A trivial content is set to `NULL`.

GEN `primpart`(GEN `x`) as above but the content is lost. (For efficiency, the content remains on the stack.)

GEN `denom_i`(GEN `x`) shallow version of `denom`.

GEN `numer_i`(GEN `x`) shallow version of `numer`.

9.10.2 Over the rationals.

long `Q_pval`(GEN `x`, GEN `p`) valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.

long `Q_pvalrem`(GEN `x`, GEN `p`, GEN `*r`) returns the valuation e at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`. The quotient x/p^e is returned in `*r`.

GEN `Q_abs`(GEN `x`) absolute value of the `t_INT` or `t_FRAC` `x`.

GEN `Qdivii`(GEN `x`, GEN `y`), assuming x and y are both of type `t_INT`, return the quotient x/y as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

GEN `Q_abs_shallow`(GEN `x`) x being a `t_INT` or a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(x)` otherwise.

GEN `Q_gcd`(GEN `x`, GEN `y`) gcd of the `t_INT` or `t_FRAC` `x` and `y`.

In the following functions, arguments belong to a $M \otimes_{\mathbf{Z}} \mathbf{Q}$ for some natural \mathbf{Z} -module M , e.g. multivariate polynomials with integer coefficients (or vectors/matrices recursively built from such objects), and an element of M is said to be *integral*. We are interested in contents, denominators, etc. with respect to this canonical integral structure; in particular, contents belong to \mathbf{Q} , denominators to \mathbf{Z} . For instance the \mathbf{Q} -content of $(1/2)xy$ is $(1/2)$, and its \mathbf{Q} -denominator is 2, whereas `content` would return $y/2$ and `denom` 1.

GEN `Q_content`(GEN `x`) the \mathbf{Q} -content of x .

GEN `Z_content`(GEN `x`) as `Q_content` but assume that all rationals are in fact `t_INTs` and return `NULL` when the content is 1. This function returns as soon as the content is found to equal 1.

GEN `Q_content_safe(GEN x)` as `Q_content`, returning NULL when the \mathbf{Q} -content is not defined (e.g. for a `t_REAL` or `t_INTMOD` component).

GEN `Q_denom(GEN x)` the \mathbf{Q} -denominator of x . Shallow function. Raises an `e_TYPE` error out when the notion is meaningless, e.g. for a `t_REAL` or `t_INTMOD` component.

GEN `Q_denom_safe(GEN x)` the \mathbf{Q} -denominator of x . Shallow function. Return NULL when the notion is meaningless.

GEN `Q_primitive_part(GEN x, GEN *c)` sets c to the \mathbf{Q} -content of x and returns x / c , which is integral.

GEN `Q_primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

GEN `vec_Q_primpart(GEN x)` as above component-wise.

GEN `Q_remove_denom(GEN x, GEN *ptd)` sets d to the \mathbf{Q} -denominator of x and returns $x * d$, which is integral. Shallow function.

GEN `Q_div_to_int(GEN x, GEN c)` returns x / c , assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

GEN `Q_mul_to_int(GEN x, GEN c)` returns $x * c$, assuming c is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

GEN `Q_muli_to_int(GEN x, GEN d)` returns $x * c$, assuming c is a `t_INT` and the result is integral.

GEN `mul_content(GEN cx, GEN cy)` cx and cy are as set by `primitive_part`: either a `GEN` or NULL representing the trivial content 1. Returns their product (either a `GEN` or NULL).

GEN `inv_content(GEN c)` c is as set by `primitive_part`: either a `GEN` or NULL representing the trivial content 1. Returns its inverse (either a `GEN` or NULL).

GEN `mul_denom(GEN dx, GEN dy)` dx and dy are as set by `Q_remove_denom`: either a `t_INT` or NULL representing the trivial denominator 1. Returns their product (either a `t_INT` or NULL).

9.11 Generic arithmetic operators.

9.11.1 Unary operators.

GEN `gneg[z](GEN x[, GEN z])` yields $-x$.

GEN `gneg_i(GEN x)` shallow function yielding $-x$.

GEN `gabs[z](GEN x[, GEN z])` yields $|x|$.

GEN `gsqr(GEN x)` creates the square of x .

GEN `ginv(GEN x)` creates the inverse of x .

9.11.2 Binary operators.

Let “*op*” be a binary operation among

op=**add**: addition ($x + y$).

op=**sub**: subtraction ($x - y$).

op=**mul**: multiplication ($x * y$).

op=**div**: division (x / y).

The names and prototypes of the functions corresponding to *op* are as follows:

GEN *gop*(GEN *x*, GEN *y*)

GEN *gopgs*(GEN *x*, long *s*)

GEN *gopsg*(long *s*, GEN *y*)

Explicitly

GEN *gadd*(GEN *x*, GEN *y*), GEN *gaddgs*(GEN *x*, long *s*), GEN *gaddsg*(long *s*, GEN *x*)

GEN *gmul*(GEN *x*, GEN *y*), GEN *gmulgs*(GEN *x*, long *s*), GEN *gmulsg*(long *s*, GEN *x*)

GEN *gsub*(GEN *x*, GEN *y*), GEN *gsubgs*(GEN *x*, long *s*), GEN *gsubsg*(long *s*, GEN *x*)

GEN *gdiv*(GEN *x*, GEN *y*), GEN *gdivgs*(GEN *x*, long *s*), GEN *gdivsg*(long *s*, GEN *x*)

GEN *gpow*(GEN *x*, GEN *y*, long *l*) creates x^y . If *y* is a `t_INT`, return *powgi*(*x*,*y*) (the precision *l* is not taken into account). Otherwise, the result is $\exp(y * \log(x))$ where exact arguments are converted to floats of precision *l* in case of need; if there is no need, for instance if *x* is a `t_REAL`, *l* is ignored. Indeed, if *x* is a `t_REAL`, the accuracy of $\log x$ is determined from the accuracy of *x*, it is no problem to multiply by *y*, even if it is an exact type, and the accuracy of the exponential is determined, exactly as in the case of the initial $\log x$.

GEN *gpowgs*(GEN *x*, long *n*) creates x^n using binary powering. To treat the special case $n = 0$, we consider *gpowgs* as a series of *gmul*, so we follow the rule of returning result which is as exact as possible given the input. More precisely, we return

- *gen_1* if *x* has type `t_INT`, `t_REAL`, `t_FRAC`, or `t_PADIC`
- *Mod*(1,*N*) if *x* is a `t_INTMOD` modulo *N*.
- *gen_1* for `t_COMPLEX`, `t_QUAD` unless one component is a `t_INTMOD`, in which case we return *Mod*(1, *N*) for a suitable *N* (the gcd of the moduli that appear).
- *FF_1*(*x*) for a `t_FFELT`.
- *qfi_1*(*x*) and *qfr_1*(*x*) for `t_QFI` and `t_QFR`.
- the identity permutation for `t_VECSMALL`.
- *Rg_get_1*(*x*) otherwise

Of course, the only practical use of this routine for $n = 0$ is to obtain the multiplicative neutral element in the base ring (or to treat marginal cases that should be special cased anyway if there is the slightest doubt about what the result should be).

GEN *powgi*(GEN *x*, GEN *y*) creates x^y , where *y* is a `t_INT`, using left-shift binary powering. The case where $y = 0$ (as all cases where *y* is small) is handled by *gpowgs*(*x*, 0).

GEN `gpowers`(GEN `x`, long `n`) returns the vector $[1, x, \dots, x^n]$.

GEN `grootsof1`(long `n`, long `prec`) returns the vector $[1, x, \dots, x^{n-1}]$, where x is the n -th root of unity $\exp(2i\pi/n)$.

GEN `gsqrpowers`(GEN `x`, long `n`) returns the vector $[x, x^4, \dots, x^{n^2}]$.

In addition we also have the obsolete forms:

```
void gaddz(GEN x, GEN y, GEN z)
```

```
void gsubz(GEN x, GEN y, GEN z)
```

```
void gmulz(GEN x, GEN y, GEN z)
```

```
void gdivz(GEN x, GEN y, GEN z)
```

9.12 Generic operators: product, powering, factorback.

To describe the following functions, we use the following private typedefs to simplify the description:

```
typedef (*F0)(void *);
typedef (*F1)(void *, GEN);
typedef (*F2)(void *, GEN, GEN);
```

They correspond to generic functions with one and two arguments respectively (the `void*` argument provides some arbitrary evaluation context).

GEN `gen_product`(GEN `v`, void `*D`, F2 `op`) Given two objects x, y , assume that `op(D, x, y)` implements an associative binary operator. If v has k entries, return

$$v[1] \text{ op } v[2] \text{ op } \dots \text{ op } v[k];$$

returns `gen_1` if $k = 0$ and a copy of $v[1]$ if $k = 1$. Use divide and conquer strategy. Leave some garbage on stack, but suitable for `gerepileupto` if `mul` is.

GEN `gen_pow`(GEN `x`, GEN `n`, void `*D`, F1 `sqr`, F2 `mul`) $n > 0$ a `t_INT`, returns x^n ; `mul(D, x, y)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D, x, x)`.

GEN `gen_powu`(GEN `x`, ulong `n`, void `*D`, F1 `sqr`, F2 `mul`) $n > 0$, returns x^n . See `gen_pow`.

GEN `gen_pow_i`(GEN `x`, GEN `n`, void `*E`, F1 `sqr`, F2 `mul`) internal variant of `gen_pow`, not memory-clean.

GEN `gen_powu_i`(GEN `x`, ulong `n`, void `*E`, F1 `sqr`, F2 `mul`) internal variant of `gen_powu`, not memory-clean.

GEN `gen_pow_fold`(GEN `x`, GEN `n`, void `*D`, F1 `sqr`, F1 `msqr`) variant of `gen_pow`, where `mul` is replaced by `msqr`, with `msqr(D, y)` returning xy^2 . In particular `D` must implicitly contain x .

GEN `gen_pow_fold_i`(GEN `x`, GEN `n`, void `*E`, F1 `sqr`, F1 `msqr`) internal variant of the function `gen_pow_fold`, not memory-clean.

GEN `gen_powu_fold`(GEN `x`, ulong `n`, void `*D`, F1 `sqr`, F1 `msqr`), see `gen_pow_fold`.

GEN `gen_powu_fold_i`(GEN `x`, ulong `n`, void `*E`, F1 `sqr`, F1 `msqr`) see `gen_pow_fold_i`.

GEN `gen_pow_init`(GEN `x`, GEN `n`, long `k`, void `*E`, GEN (`*sqr`)(void*, GEN), GEN (`*mul`)(void*, GEN, GEN)) Return a table `R` that can be used with `gen_pow_table` to compute the powers of x up to n . The table is of size $2^k \log_2(n)$.

GEN `gen_pow_table`(GEN `R`, GEN `n`, void `*E`, GEN (`*one`)(void*), GEN (`*mul`)(void*, GEN, GEN))

Return x^n , where R is as given by `gen_pow_init(x,m,k,E,sqr,mul)` for some integer $m \geq n$.

GEN `gen_powers`(GEN `x`, long `n`, long `usesqr`, void `*D`, F1 `sqr`, F2 `mul`, F0 `one`) returns $[x^0, \dots, x^n]$ as a `t_VEC`; `mul(D, x, y)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D, x, x)`; `one` returns the monoid unit. The flag `usesqr` should be set to 1 if squaring are faster than multiplication by x .

GEN `gen_factorback`(GEN `L`, GEN `e`, F2 `mul`, F2 `pow`, void `*D`) generic form of `factorback`. The pair $[L, e]$ is of the form

- `[fa, NULL]`, `fa` a two-column factorization matrix: expand it.
- `[v, NULL]`, `v` a vector of objects: return their product.
- or `[v, e]`, `v` a vector of objects, `e` a vector of integral exponents: return the product of the $v[i]^{e[i]}$.

`mul(D, x, y)` and `pow(D, x, n)` return xy and x^n respectively.

9.13 Matrix and polynomial norms.

This section concerns only standard norms of \mathbf{R} and \mathbf{C} vector spaces, not algebraic norms given by the determinant of some multiplication operator. We have already seen type-specific functions like `ZM_supnorm` or `RgM_fpnorml2` and limit ourselves to generic functions assuming nothing about their `GEN` argument; these functions allow the following scalar types: `t_INT`, `t_FRAC`, `t_REAL`, `t_COMPLEX`, `t_QUAD` and are defined recursively (in terms of norms of their components) for the following “container” types: `t_POL`, `t_VEC`, `t_COL` and `t_MAT`. They raise an error if some other type appears in the argument.

GEN `gnorml2`(GEN `x`) The norm of a scalar is the square of its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the *square* of the usual L^2 norm. In most applications, the missing square root computation can be skipped.

GEN `gnorml1`(GEN `x`, long `prec`) The norm of a scalar is its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the usual L^1 norm. One must include a real precision `prec` in case the inputs include `t_COMPLEX` or `t_QUAD` with exact rational components: a square root must be computed and we must choose an accuracy.

GEN `gnorml1_fake`(GEN `x`) as `gnorml1`, except that the norm of a `t_QUAD` $x + wy$ or `t_COMPLEX` $x + Iy$ is defined as $|x| + |y|$, where we use the ordinary real absolute value. This is still a norm of \mathbf{R} vector spaces, which is easier to compute than `gnorml1` and can often be used in its place.

GEN `gsupnorm`(GEN `x`, long `prec`) The norm of a scalar is its complex modulus, the norm of a recursive type is the max of the norms of its components. A precision `prec` must be included for the same reason as in `gnorml1`.

`void gsupnorm_aux(GEN x, GEN *m, GEN *m2, long prec)` is the low-level function underlying `gsupnorm`, used as follows:

```
GEN m = NULL, m2 = NULL;
gsupnorm_aux(x, &m, &m2);
```

After the call, the sup norm of x is the min of m and the square root of $m2$; one or both of m , $m2$ may be NULL, in which case it must be omitted. You may initially set m and $m2$ to non-NULL values, in which case, the above procedure yields the max of (the initial) m , the square root of (the initial) $m2$, and the sup norm of x .

The strange interface is due to the fact that $|z|^2$ is easier to compute than $|z|$ for a `t_QUAD` or `t_COMPLEX` z : $m2$ is the max of those $|z|^2$, and m is the max of the other $|z|$.

9.14 Substitution and evaluation.

`GEN gsubst(GEN x, long v, GEN y)` substitutes the object y into x for the variable number v .

`GEN poleval(GEN q, GEN x)` evaluates the `t_POL` or `t_RFRAC` q at x . For convenience, a `t_VEC` or `t_COL` is also recognized as the `t_POL` `gtovecrev(q)`.

`GEN RgX_cxeval(GEN T, GEN x, GEN xi)` evaluate the `t_POL` T at x via Horner's scheme. If xi is not NULL it must be equal to $1/x$ and we evaluate $x^{\deg T}T(1/x)$ instead. This is useful when $|x| > 1$ is a `t_REAL` or an inexact `t_COMPLEX` and T has "balanced" coefficients, since the evaluation becomes numerically stable.

`GEN RgX_RgM_eval(GEN q, GEN x)` evaluates the `t_POL` q at the square matrix x .

`GEN RgX_RgMV_eval(GEN f, GEN V)` returns the evaluation $f(x)$, assuming that V was computed by `FpXQ_powers(x, n)` for some $n > 1$.

`GEN qfeval(GEN q, GEN x)` evaluates the quadratic form q (symmetric matrix) at x (column vector of compatible dimensions).

`GEN qfevalb(GEN q, GEN x, GEN y)` evaluates the polar bilinear form attached to the quadratic form q (symmetric matrix) at x, y (column vectors of compatible dimensions).

`GEN hqfeval(GEN q, GEN x)` evaluates the Hermitian form q (a Hermitian complex matrix) at x .

`GEN qf_apply_RgM(GEN q, GEN M)` q is a symmetric $n \times n$ matrix, M an $n \times k$ matrix, return $M'qM$.

`GEN qf_apply_ZM(GEN q, GEN M)` as above assuming that both q and M have integer entries.

Chapter 10: Miscellaneous mathematical functions

10.1 Fractions.

GEN `absfrac(GEN x)` returns the absolute value of the `t_FRAC` x .

GEN `absfrac_shallow(GEN x)` x being a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(x)` otherwise.

GEN `sqrfrac(GEN x)` returns the square of the `t_FRAC` x .

10.2 Binomials.

GEN `binomial(GEN x, long k)`

GEN `binomialuu(ulong n, ulong k)`

GEN `vecbinomial(long n)`, which returns a vector v with $n + 1$ `t_INT` components such that $v[k + 1] = \text{binomial}(n, k)$ for k from 0 up to n .

10.3 Real numbers.

GEN `R_abs(GEN x)` x being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns $|x|$.

GEN `R_abs_shallow(GEN x)` x being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns a shallow copy of $|x|$, in particular returns x itself when $x \geq 0$, and `gneg(x)` otherwise.

GEN `modRr_safe(GEN x, GEN y)` let x be a `t_INT`, a `t_REAL` or `t_FRAC` and let y be a `t_REAL`. Return $x \% y$ unless the input accuracy is insufficient to compute the floor or x/y in which case we return `NULL`.

10.4 Complex numbers.

GEN `gimag`(GEN `x`) returns a copy of the imaginary part of `x`.

GEN `greal`(GEN `x`) returns a copy of the real part of `x`. If `x` is a `t_QUAD`, returns the coefficient of 1 in the “canonical” integral basis $(1, \omega)$.

GEN `gconj`(GEN `x`) returns $\text{greal}(x) - 2\text{gimag}(x)$, which is the ordinary complex conjugate except for a real `t_QUAD`.

GEN `imag_i`(GEN `x`), shallow variant of `gimag`.

GEN `real_i`(GEN `x`), shallow variant of `greal`.

GEN `conj_i`(GEN `x`), shallow variant of `gconj`.

GEN `mulreal`(GEN `x`, GEN) returns the real part of xy ; x, y have type `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`. See also `RgM_mulreal`.

GEN `cxnorm`(GEN `x`) norm of the `t_COMPLEX` x (modulus squared).

GEN `cxexpm1`(GEN `x`) returns $\exp(x) - 1$, for a `t_COMPLEX` x .

int `cx_approx_equal`(GEN `a`, GEN `b`) test whether (`t_INT`, `t_FRAC`, `t_REAL`, or `t_COMPLEX` of those) a and b are approximately equal. This returns 1 if and only if the division by $a - b$ would produce a division by 0 (which is a less stringent test than testing whether $a - b$ evaluates to 0).

10.5 Quadratic numbers and binary quadratic forms.

GEN `quad_disc`(GEN `x`) returns the discriminant of the `t_QUAD` x . Not stack-clean but suitable for `gerepileupto`.

GEN `quadnorm`(GEN `x`) norm of the `t_QUAD` x .

GEN `qfb_disc`(GEN `x`) returns the discriminant of the `t_QFI` or `t_QFR` x .

GEN `qfb_disc3`(GEN `x`, GEN `y`, GEN `z`) returns $y^2 - 4xz$ assuming all inputs are `t_INTs`. Not stack-clean.

GEN `qfb_apply_ZM`(GEN `q`, GEN `g`) returns $q \circ g$.

GEN `qfbforms`(GEN `D`) given a discriminant $D < 0$, return the list of reduced forms of discriminant D as `t_VECSMALL` with 3 components. The primitive forms in the list enumerate the class group of the quadratic order of discriminant D ; if D is fundamental, all returned forms are automatically primitive.

10.6 Polynomials.

GEN truecoef(GEN x , long n) returns `polcoef(x,n,-1)`, i.e. the coefficient of the term of degree n in the main variable. This is a safe but expensive function that must *copy* its return value so that it be `gerepile`-safe. Use `polcoef_i` for a fast internal variant.

GEN polcoef_i(GEN x , long n , long v) internal shallow function. Rewrite x as a Laurent polynomial in the variable v and returns its coefficient of degree n (`gen_0` if this falls outside the coefficient array). Allow `t_POL`, `t_SER`, `t_RFRAC` and scalars.

long degree(GEN x) returns `poldegree(x,-1)`, the degree of x with respect to its main variable, with the usual meaning if the leading coefficient of x is non-zero. If the sign of x is 0, this function always returns -1 . Otherwise, we return the index of the leading coefficient of x , i.e. the coefficient of largest index stored in x . For instance the “degrees” of

```
0. E-38 * x^4 + 0.E-19 * x + 1
Mod(0,2) * x^0    \\ sign is 0 !
```

are 4 and -1 respectively.

long degpol(GEN x) is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL` x with respect to its main variable, *if* its leading coefficient is non-zero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. `Mod(0,2)`). If x has no coefficients (rational 0 polynomial), its length is 2 and we return the expected -1 .

GEN characteristic(GEN x) returns the characteristic of the base ring over which the polynomial is defined (as defined by `t_INTMOD` and `t_FFELT` components). The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

GEN residual_characteristic(GEN x) returns a kind of “residual characteristic” of the base ring over which the polynomial is defined. This is defined as the gcd of all moduli `t_INTMODs` occurring in the structure, as well as primes p arising from `t_PADICs` or `t_FFELTs`. The function raises an exception if incompatible primes arise from `t_FFELT` and `t_PADIC` components. Shallow function.

GEN resultant(GEN x , GEN y) resultant of x and y , with respect to the main variable of highest priority. Uses either the subresultant algorithm (generic case), a modular algorithm (inputs in $\mathbb{Q}[X]$) or Sylvester’s matrix (inexact inputs).

GEN resultant2(GEN x , GEN y) resultant of x and y , with respect to the main variable of highest priority. Computes the determinant of Sylvester’s matrix.

GEN cleanroots(GEN x , long $prec$) returns the complex roots of the complex polynomial x (with coefficients `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX` of the above). The roots are returned as `t_REAL` or `t_COMPLEX` of `t_REALs` of precision $prec$ (guaranteeing a non-0 imaginary part). See `QX_complex_roots`.

double fujiwara_bound(GEN x) return a quick upper bound for the logarithm in base 2 of the modulus of the largest complex roots of the polynomial x (complex coefficients).

double fujiwara_bound_real(GEN x , long $sign$) return a quick upper bound for the logarithm in base 2 of the absolute value of the largest real root of sign $sign$ (1 or -1), for the polynomial x (real coefficients).

GEN polmod_to_embed(GEN x , long $prec$) return the vector of complex embeddings of the `t_POLMOD` x (with complex coefficients). Shallow function, simple complex variant of `conjvec`.

10.7 Power series.

GEN `sertoser`(GEN `x`, long `prec`) return the `t_SER` `x` truncated or extended (with zeros) to `prec` terms. Shallow function, assume that `prec` ≥ 0 .

GEN `derivser`(GEN `x`) returns the derivative of the `t_SER` `x` with respect to its main variable.

GEN `integser`(GEN `x`) returns the primitive of the `t_SER` `x` with respect to its main variable.

GEN `truecoef`(GEN `x`, long `n`) returns `polcoef(x,n,-1)`, i.e. the coefficient of the term of degree `n` in the main variable. This is a safe but expensive function that must *copy* its return value so that it be `gerepile`-safe. Use `polcoef_i` for a fast internal variant.

GEN `ser_unscale`(GEN `P`, GEN `h`) return $P(hx)$, not memory clean.

GEN `ser_normalize`(GEN `x`) divide `x` by its “leading term” so that the series is either 0 or equal to $t^v(1 + O(t))$. Shallow function if the “leading term” is 1.

int `ser_isexactzero`(GEN `x`) return 1 if `x` is a zero series, all of whose known coefficients are exact zeroes; this implies that `sign(x) = 0` and `lg(x) ≤ 3` .

GEN `ser_inv`(GEN `x`) return the inverse of the `t_SER` `x` using Newton iteration. This is in general slower than `ginv` unless the precision is huge (hundreds of terms, where the threshold depends strongly on the base field).

10.8 Functions to handle `t_FFELT`.

These functions define the public interface of the `t_FFELT` type to use in generic functions. However, in specific functions, it is better to use the functions class `FpXQ` and/or `F1xq` as appropriate.

GEN `FF_p`(GEN `a`) returns the characteristic of the definition field of the `t_FFELT` element `a`.

long `FF_f`(GEN `a`) returns the dimension of the definition field over its prime field; the cardinality of the dimension field is thus p^f .

GEN `FF_p_i`(GEN `a`) shallow version of `FF_p`.

GEN `FF_q`(GEN `a`) returns the cardinality of the definition field of the `t_FFELT` element `a`.

GEN `FF_mod`(GEN `a`) returns the polynomial (with reduced `t_INT` coefficients) defining the finite field, in the variable used to display `a`.

GEN `FF_gen`(GEN `a`) returns the standard generator of the definition field of the `t_FFELT` element `a`, see `ffgen`, that is $x \pmod{T}$ where T is the polynomial over the prime field that define the finite field.

GEN `FF_to_FpXQ`(GEN `a`) converts the `t_FFELT` `a` to a polynomial P with reduced `t_INT` coefficients such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g .

GEN `FF_to_FpXQ_i`(GEN `a`) shallow version of `FF_to_FpXQ`.

GEN `FF_to_F2xq`(GEN `a`) converts the `t_FFELT` `a` to a `F2x` P such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g . This only work if the characteristic is 2.

GEN `FF_to_F2xq_i`(GEN `a`) shallow version of `FF_to_F2xq`.

GEN `FF_to_Flxq`(GEN `a`) converts the `t_FFELT` `a` to a `Flx P` such that $a = P(g)$ where g is the generator of the finite field returned by `ffgen`, in the variable used to display g . This only work if the characteristic is small enough.

GEN `FF_to_Flxq_i`(GEN `a`) shallow version of `FF_to_Flxq`.

GEN `p_to_FF`(GEN `p`, long `v`) returns a `t_FFELT` equal to 1 in the finite field $\mathbf{Z}/p\mathbf{Z}$. Useful for generic code that wants to handle (inefficiently) $\mathbf{Z}/p\mathbf{Z}$ as if it were not a prime field.

GEN `Tp_to_FF`(GEN `T`, GEN `p`) returns a `t_FFELT` equal to 1 in the finite field $\mathbf{F}_p/(T)$, where T is a `ZX`, assumed to be irreducible modulo p , or `NULL` in which case the routine acts as `p_to_FF(p,0)`. No checks.

GEN `Fq_to_FF`(GEN `x`, GEN `ff`) returns a `t_FFELT` equal to x in the finite field defined by the `t_FFELT` `ff`, where x is an `Fq` (either a `t_INT` or a `ZX`: a `t_POL` with `t_INT` coefficients). No checks.

GEN `FqX_to_FFX`(GEN `x`, GEN `ff`) given an `FqX` x , return the polynomial with `t_FFELT` coefficients obtained by applying `Fq_to_FF` coefficientwise. No checks, and no normalization if the leading coefficient maps to 0.

GEN `FF_1`(GEN `a`) returns the unity in the definition field of the `t_FFELT` element `a`.

GEN `FF_zero`(GEN `a`) returns the zero element of the definition field of the `t_FFELT` element `a`.

int `FF_equal0`(GEN `a`) returns 1 if the `t_FFELT` `a` is equal to 0 else returns 0.

int `FF_equal1`(GEN `a`) returns 1 if the `t_FFELT` `a` is equal to 1 else returns 0.

int `FF_equalm1`(GEN `a`) returns -1 if the `t_FFELT` `a` is equal to 1 else returns 0.

int `FF_equal`(GEN `a`, GEN `b`) return 1 if the `t_FFELT` `a` and `b` have the same definition field and are equal, else 0.

int `FF_samefield`(GEN `a`, GEN `b`) return 1 if the `t_FFELT` `a` and `b` have the same definition field, else 0.

int `Rg_is_FF`(GEN `c`, GEN `*ff`) to be called successively on many objects, setting `*ff = NULL` (unset) initially. Returns 1 as long as c is a `t_FFELT` defined over the same field as `*ff` (setting `*ff = c` if unset), and 0 otherwise.

int `RgC_is_FFC`(GEN `x`, GEN `*ff`) apply `Rg_is_FF` successively to all components of the `t_VEC` or `t_COL` x . Return 0 if one call fails, and 1 otherwise.

int `RgM_is_FFM`(GEN `x`, GEN `*ff`) apply `Rg_is_FF` to all components of the `t_MAT`. Return 0 if one call fails, and 1 otherwise.

GEN `FF_add`(GEN `a`, GEN `b`) returns $a + b$ where `a` and `b` are `t_FFELT` having the same definition field.

GEN `FF_Z_add`(GEN `a`, GEN `x`) returns $a + x$, where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

GEN `FF_Q_add`(GEN `a`, GEN `x`) returns $a + x$, where `a` is a `t_FFELT`, and `x` is a `t_RFRAC`, the computation being performed in the definition field of `a`.

GEN `FF_sub`(GEN `a`, GEN `b`) returns $a - b$ where `a` and `b` are `t_FFELT` having the same definition field.

GEN `FF_mul`(GEN `a`, GEN `b`) returns ab where `a` and `b` are `t_FFELT` having the same definition field.

GEN `FF_Z_mul`(GEN `a`, GEN `b`) returns ab , where `a` is a `t_FFELT`, and `b` is a `t_INT`, the computation being performed in the definition field of `a`.

GEN `FF_div`(GEN `a`, GEN `b`) returns a/b where `a` and `b` are `t_FFELT` having the same definition field.

GEN `FF_neg`(GEN `a`) returns $-a$ where `a` is a `t_FFELT`.

GEN `FF_neg_i`(GEN `a`) shallow function returning $-a$ where `a` is a `t_FFELT`.

GEN `FF_inv`(GEN `a`) returns a^{-1} where `a` is a `t_FFELT`.

GEN `FF_sqr`(GEN `a`) returns a^2 where `a` is a `t_FFELT`.

GEN `FF_mul2n`(GEN `a`, long `n`) returns a^{2^n} where `a` is a `t_FFELT`.

GEN `FF_pow`(GEN `a`, GEN `n`) returns a^n where `a` is a `t_FFELT` and `n` is a `t_INT`.

GEN `FF_Frobenius`(GEN `a`, GEN `n`) returns a^{p^n} where `a` is a `t_FFELT` `n` is a `t_INT`, and p is the characteristic of the definition field of `a`.

GEN `FF_Z_Z_muldiv`(GEN `a`, GEN `x`, GEN `y`) returns ay/z , where `a` is a `t_FFELT`, and `x` and `y` are `t_INT`, the computation being performed in the definition field of `a`.

GEN `Z_FF_div`(GEN `x`, GEN `a`) return x/a where `a` is a `t_FFELT`, and `x` is a `t_INT`, the computation being performed in the definition field of `a`.

GEN `FF_norm`(GEN `a`) returns the norm of the `t_FFELT` `a` with respect to its definition field.

GEN `FF_trace`(GEN `a`) returns the trace of the `t_FFELT` `a` with respect to its definition field.

GEN `FF_conjvec`(GEN `a`) returns the vector of conjugates $[a, a^p, a^{p^2}, \dots, a^{p^{n-1}}]$ where the `t_FFELT` `a` belong to a field with p^n elements.

GEN `FF_charpoly`(GEN `a`) returns the characteristic polynomial) of the `t_FFELT` `a` with respect to its definition field.

GEN `FF_minpoly`(GEN `a`) returns the minimal polynomial of the `t_FFELT` `a`.

GEN `FF_sqrt`(GEN `a`) returns an `t_FFELT` `b` such that $a = b^2$ if it exist, where `a` is a `t_FFELT`.

long `FF_issquareall`(GEN `x`, GEN `*pt`) returns 1 if `x` is a square, and 0 otherwise. If `x` is indeed a square, set `pt` to its square root.

long `FF_issquare`(GEN `x`) returns 1 if `x` is a square and 0 otherwise.

long `FF_isplayer`(GEN `x`, GEN `K`, GEN `*pt`) Given K a positive integer, returns 1 if `x` is a K -th power, and 0 otherwise. If `x` is indeed a K -th power, set `pt` to its K -th root.

GEN `FF_sqrtn`(GEN `a`, GEN `n`, GEN `*zn`) returns an n -th root of `a` if it exist. If `zn` is non-NULL set it to a primitive n -th root of the unity.

GEN `FF_log`(GEN `a`, GEN `g`, GEN `o`) the `t_FFELT` `g` being a generator for the definition field of the `t_FFELT` `a`, returns a `t_INT` `e` such that $a^e = g$. If `e` does not exists, the result is currently undefined. If `o` is not NULL it is assumed to be a factorization of the multiplicative order of `g` (as set by `FF_primroot`)

GEN `FF_order`(GEN `a`, GEN `o`) returns the order of the `t_FFELT` `a`. If `o` is non-NULL, it is assumed that `o` is a multiple of the order of `a`.

GEN `FF_primroot`(GEN `a`, GEN `*o`) returns a generator of the multiplicative group of the definition field of the `t_FFELT` `a`. If `o` is not `NULL`, set it to the factorization of the order of the primitive root (to speed up `FF_log`).

GEN `FF_map`(GEN `m`, GEN `a`) returns $A(m)$ where `A=a.pol` assuming a and m belongs to fields having the same characteristic.

10.8.1 FFX.

The functions in this sections take polynomial arguments and a `t_FFELT` a . The coefficients of the polynomials must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with a .

GEN `FFX_mul`(GEN `P`, GEN `Q`, GEN `a`) returns the product of the polynomials `P` and `Q` defined over the definition field of the `t_FFELT` a .

GEN `FFX_sqr`(GEN `P`, GEN `a`) returns the square of the polynomial `P` defined over the definition field of the `t_FFELT` a .

GEN `FFX_rem`(GEN `P`, GEN `Q`, GEN `a`) returns the remainder of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT` a .

GEN `FFX_isplayer`(GEN `P`, `ulong` `k`, GEN `a`, GEN `*py`) return 1 if the FFX P is a k -th power, 0 otherwise, where `P` is defined over the definition field of the `t_FFELT` a . If `py` is not `NULL`, set it to g such that $g^k = f$.

GEN `FFX_factor`(GEN `f`, GEN `a`) returns the factorization of the univariate polynomial `f` over the definition field of the `t_FFELT` a . The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with a .

GEN `FFX_factor_squarefree`(GEN `f`, GEN `a`) returns the squarefree factorization of the univariate polynomial `f` over the definition field of the `t_FFELT` a . This is a vector $[u_1, \dots, u_k]$ of pairwise coprime FFX such that $u_k \neq 1$ and $f = \prod u_i^i$.

GEN `FFX_ddf`(GEN `f`, GEN `a`) assuming that f is squarefree, returns the distinct degree factorization of f modulo p . The returned value `v` is a `t_VEC` with two components: `F=v[1]` is a vector of (FFX) factors, and `E=v[2]` is a `t_VECSMALL`, such that f is equal to the product of the `F[i]` and each `F[i]` is a product of irreducible factors of degree `E[i]`.

GEN `FFX_degfact`(GEN `f`, GEN `a`), as `FFX_factor`, but the degrees of the irreducible factors are returned instead of the factors themselves (as a `t_VECSMALL`).

GEN `FFX_roots`(GEN `f`, GEN `a`) returns the roots (`t_FFELT`) of the univariate polynomial `f` over the definition field of the `t_FFELT` a . The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with a .

GEN `FFX_preimage`(GEN `F`, GEN `x`, GEN `a`) returns $P\%F$ where `P=x.pol` assuming a and x belongs to fields having the same characteristic, and that the coefficients of F belong to the definition field of a .

10.8.2 FFM.

GEN FFM_FFC_gauss(GEN M, GEN C, GEN ff) given a matrix M ($\mathfrak{t_MAT}$) and a column vector C ($\mathfrak{t_COL}$) over the finite field given by ff ($\mathfrak{t_FFELT}$) such that M is invertible, return the unique column vector X such that $MX = C$.

GEN FFM_FFC_invimage(GEN M, GEN C, GEN ff) given a matrix M ($\mathfrak{t_MAT}$) and a column vector C ($\mathfrak{t_COL}$) over the finite field given by ff ($\mathfrak{t_FFELT}$), return a column vector X such that $MX = C$, or NULL if no such vector exists.

GEN FFM_FFC_mul(GEN M, GEN C, GEN ff) returns the product of the matrix M ($\mathfrak{t_MAT}$) and the column vector C ($\mathfrak{t_COL}$) over the finite field given by ff ($\mathfrak{t_FFELT}$).

GEN FFM_deplin(GEN M, GEN ff) returns a non-zero vector ($\mathfrak{t_COL}$) in the kernel of the matrix M over the finite field given by ff, or NULL if no such vector exists.

GEN FFM_det(GEN M, GEN ff) returns the determinant of the matrix M over the finite field given by ff.

GEN FFM_gauss(GEN M, GEN N, GEN ff) given two matrices M and N ($\mathfrak{t_MAT}$) over the finite field given by ff ($\mathfrak{t_FFELT}$) such that M is invertible, return the unique matrix X such that $MX = N$.

GEN FFM_image(GEN M, GEN ff) returns a matrix whose columns span the image of the matrix M over the finite field given by ff.

GEN FFM_indexrank(GEN M, GEN ff) given a matrix M of rank r over the finite field given by ff, returns a vector with two $\mathfrak{t_VECSMALL}$ components y and z containing r row and column indices, respectively, such that the $r \times r$ -matrix formed by the $M[i, j]$ for i in y and j in z is invertible.

GEN FFM_inv(GEN M, GEN ff) returns the inverse of the square matrix M over the finite field given by ff, or NULL if M is not invertible.

GEN FFM_invimage(GEN M, GEN N, GEN ff) given two matrices M and N ($\mathfrak{t_MAT}$) over the finite field given by ff ($\mathfrak{t_FFELT}$), return a matrix X such that $MX = N$, or NULL if no such matrix exists.

GEN FFM_ker(GEN M, GEN ff) returns the kernel of the $\mathfrak{t_MAT}$ M over the finite field given by the $\mathfrak{t_FFELT}$ ff.

GEN FFM_mul(GEN M, GEN N, GEN ff) returns the product of the matrices M and N ($\mathfrak{t_MAT}$) over the finite field given by ff ($\mathfrak{t_FFELT}$).

long FFM_rank(GEN M, GEN ff) returns the rank of the matrix M over the finite field given by ff.

GEN FFM_suppl(GEN M, GEN ff) given a matrix M over the finite field given by ff whose columns are linearly independent, returns a square invertible matrix whose first columns are those of M.

10.8.3 FFXQ.

GEN FFXQ_mul(GEN P, GEN Q, GEN T, GEN a) returns the product of the polynomials P and Q modulo the polynomial T, where P, Q and T are defined over the definition field of the $\mathfrak{t_FFELT}$ a.

GEN FFXQ_sqr(GEN P, GEN T, GEN a) returns the square of the polynomial P modulo the polynomial T, where P and T are defined over the definition field of the $\mathfrak{t_FFELT}$ a.

GEN FFXQ_inv(GEN P, GEN Q, GEN a) returns the inverse of the polynomial P modulo the polynomial Q, where P and Q are defined over the definition field of the $\mathfrak{t_FFELT}$ a.

10.9 Transcendental functions.

The following two functions are only useful when interacting with `gp`, to manipulate its internal default precision (expressed as a number of decimal digits, not in words as used everywhere else):

`long getrealprecision(void)` returns `realprecision`.

`long setrealprecision(long n, long *prec)` sets the new `realprecision` to `n`, which is returned. As a side effect, set `prec` to the corresponding number of words `ndec2prec(n)`.

10.9.1 Transcendental functions with `t_REAL` arguments.

In the following routines, x is assumed to be a `t_REAL` and the result is a `t_REAL` (sometimes a `t_COMPLEX` with `t_REAL` components), with the largest accuracy which can be deduced from the input. The naming scheme is inconsistent here, since we sometimes use the prefix `mp` even though `t_INT` inputs are forbidden:

`GEN sqrtr(GEN x)` returns the square root of x .

`GEN cbrtr(GEN x)` returns the real cube root of x .

`GEN qrtnr(GEN x, long n)` returns the n -th root of x , assuming $n \geq 1$ and $x \geq 0$.

`GEN qrtnr_abs(GEN x, long n)` returns the n -th root of $|x|$, assuming $n \geq 1$ and $x \neq 0$.

`GEN mpcos[z](GEN x[, GEN z])` returns $\cos(x)$.

`GEN mpsin[z](GEN x[, GEN z])` returns $\sin(x)$.

`GEN mplog[z](GEN x[, GEN z])` returns $\log(x)$. We must have $x > 0$ since the result must be a `t_REAL`. Use `glog` for the general case, where you want such computations as $\log(-1) = I$.

`GEN mpexp[z](GEN x[, GEN z])` returns $\exp(x)$.

`GEN mpexpm1(GEN x)` returns $\exp(x) - 1$, but is more accurate than `subrs(mpexp(x), 1)`, which suffers from catastrophic cancellation if $|x|$ is very small.

`void mpsincosm1(GEN x, GEN *s, GEN *c)` sets s and c to $\sin(x)$ and $\cos(x) - 1$ respectively, where x is a `t_REAL`; the latter is more accurate than `subrs(mpcos(y), 1)`, which suffers from catastrophic cancellation if $|x|$ is very small.

`GEN mpveceint1(GEN C, GEN eC, long n)` as `veceint1`; assumes that $C > 0$ is a `t_REAL` and that `eC` is `NULL` or `mpexp(C)`.

`GEN mpeint1(GEN x, GEN expx)` returns `eint1(x)`, for a `t_REAL` $x \geq 0$, assuming that `expx` is `mpexp(x)`.

`GEN mplambertW(GEN y)` solution x of the implicit equation $x \exp(x) = y$, for $y > 0$ a `t_REAL`.

Useful low-level functions which *disregard* the sign of x :

`GEN sqrtr_abs(GEN x)` returns $\sqrt{|x|}$ assuming $x \neq 0$.

`GEN cbrtr_abs(GEN x)` returns $|x|^{1/3}$ assuming $x \neq 0$.

`GEN exp1r_abs(GEN x)` returns $\exp(|x|) - 1$, assuming $x \neq 0$.

`GEN logr_abs(GEN x)` returns $\log(|x|)$, assuming $x \neq 0$.

10.9.2 Other complex transcendental functions.

GEN `szeta(long s, long prec)` returns the value of Riemann's zeta function at the (possibly negative) integer $s \neq 1$, in relative accuracy `prec`.

GEN `veczeta(GEN a, GEN b, long N, long prec)` returns in a vector all the $\zeta(aj + b)$, where $j = 0, 1, \dots, N - 1$, where a and b are real numbers (of arbitrary type, although `t_INT` is treated more efficiently) and $b > 1$. Assumes that $N \geq 1$.

GEN `ggamma1m1(GEN x, long prec)` return $\Gamma(1 + x) - 1$ assuming $|x| < 1$. Guard against cancellation when x is small.

A few variants on sin and cos:

void `mpsincos(GEN x, GEN *s, GEN *c)` sets s and c to $\sin(x)$ and $\cos(x)$ respectively, where x is a `t_REAL`

GEN `expIr(GEN x)` returns $\exp(ix)$, where x is a `t_REAL`. The return type is `t_COMPLEX` unless the imaginary part is equal to 0 to the current accuracy (its sign is 0).

GEN `expIxy(GEN x, GEN y, long prec)` returns $\exp(ixy)$. Efficient when x is real and y pure imaginary.

void `gsincos(GEN x, GEN *s, GEN *c, long prec)` general case.

GEN `rootsof1_cx(GEN d, long prec)` return $e(1/d)$ at precision `prec`, $e(x) = \exp(2i\pi x)$.

GEN `rootsof1u_cx(ulong d, long prec)` return $e(1/d)$ at precision `prec`.

GEN `rootsof1q_cx(long a, long b, long prec)` return $e(a/b)$ at precision `prec`.

GEN `rootsof1powinit(long a, long b, long prec)` precompute b -th roots of 1 for `rootsof1pow`, i.e. to later compute $e(ac/b)$ for varying c .

GEN `rootsof1pow(GEN T, long c)` given $T = \text{rootsof1powinit}(a, b, \text{prec})$, return $e(ac/b)$.

A generalization of `affrr_fixlg`

GEN `affc_fixlg(GEN x, GEN res)` assume `res` was allocated using `cgetc`, and that x is either a `t_REAL` or a `t_COMPLEX` with `t_REAL` components. Assign x to `res`, first shortening the components of `res` if needed (in a `gerepile-safe` way). Further convert `res` to a `t_REAL` if x is a `t_REAL`.

GEN `trans_eval(const char *fun, GEN (*f) (GEN, long), GEN x, long prec)` evaluate the transcendental function f (named "`fun`" at the argument x and precision `prec`). This is a quick way to implement a transcendental function to be made available under GP, starting from a C function handling only `t_REAL` and `t_COMPLEX` arguments. This routine first converts x to a suitable type:

- `t_INT/t_FRAC` to `t_REAL` of precision `prec`, `t_QUAD` to `t_REAL` or `t_COMPLEX` of precision `prec`.

- `t_POLMOD` to a `t_COL` of complex embeddings (as in `conjvec`)

Then evaluates the function at `t_VEC`, `t_COL`, `t_MAT` arguments coefficientwise.

10.9.3 Modular functions.

GEN `cxredsl2`(GEN `z`, GEN `*g`) given t a `t_COMPLEX` belonging to the upper half-plane, find $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma \cdot z$ belongs to the standard fundamental domain and set `*g` to γ .

GEN `cxredsl2_i`(GEN `z`, GEN `*g`, GEN `*czd`) as `cxredsl2`; also sets `*czd` to $cz+d$, if $\gamma = [a, b; c, d]$.

GEN `cxEk`(GEN `tau`, long `k`, long `prec`) returns $E_k(\tau)$ by direct evaluation of $1 + 2/\zeta(1-k) \sum_n n^{k-1} q^n / (1 - q^n)$, $q = e(\tau)$. Assume that $\Im \tau > 0$ and k even. Very slow unless τ is already reduced modulo $\mathrm{SL}_2(\mathbf{Z})$. Not `gerepile-clean` but suitable for `gerepileupto`.

10.9.4 Transcendental functions with `t_PADIC` arguments.

GEN `Qp_exp`(GEN `x`) shortcut for `gexp(x, /*ignored*/prec)`

GEN `Qp_gamma`(GEN `x`) shortcut for `ggamma(x, /*ignored*/prec)`

GEN `Qp_log`(GEN `x`) shortcut for `glog(x, /*ignored*/prec)`

GEN `Qp_sqrt`(GEN `x`) shortcut for `gsqrt(x, /*ignored*/prec)` Return NULL if x is not a square.

GEN `Qp_sqrtn`(GEN `x`, GEN `n`, GEN `*z`) shortcut for `gsqrtn(x, n, z, /*ignored*/prec)`. Return NULL if x is not an n -th power.

GEN `Qp_agm2_sequence`(GEN `a1`, GEN `b1`) assume $a_1/b_1 = 1 \pmod p$ if p odd and $\pmod{2^4}$ if $p = 2$. Let $A_1 = a_1/p^v$ and $B_1 = b_1/p^v$ with $v = v_p(a_1) = v_p(b_1)$; let further $A_{n+1} = (A_n + B_n + 2B_{n+1})/4$, $B_{n+1} = B_n \sqrt{A_n/B_n}$ (the square root of $A_n B_n$ congruent to $B_n \pmod p$) and $R_n = p^v (A_n - B_n)$. We stop when R_n is 0 at the given p -adic accuracy. This function returns in a triplet `t_VEC` the three sequences (A_n) , (B_n) and (R_n) , corresponding to a sequence of 2-isogenies on the Tate curve $y^2 = x(x - a_1)(x + a_1 - b_1)$. The common limit of A_n and B_n is the $M_2(a_1, b_1)$, the square of the p -adic AGM of $\sqrt{a_1}$ and $\sqrt{b_1}$. This is given by `ellQp_Ei` and is used by corresponding ascending and descending p -adic Landen transforms:

`void Qp_ascending_Landen(GEN ABR, GEN *ptx, GEN *pty)`

`void Qp_descending_Landen(GEN ABR, GEN *ptx, GEN *pty)`

10.9.5 Cached constants.

The cached constant is returned at its current precision, which may be larger than `prec`. One should always use the `mpxxx` variant: `mppi`, `mpeuler`, or `mplog2`.

GEN `consteuler`(long `prec`) precomputes Euler-Mascheroni's constant at precision `prec`.

GEN `constcatalan`(long `prec`) precomputes Catalan's constant at precision `prec`.

GEN `constpi`(long `prec`) precomputes π at precision `prec`.

GEN `constlog2`(long `prec`) precomputes $\log(2)$ at precision `prec`.

`void mpbern`(long `n`, long `prec`) precomputes the n even Bernoulli numbers B_2, \dots, B_{2n} as `t_FRAC` or `t_REALs` of precision `prec`. For any $2 \leq k \leq 2n$, if a floating point approximation of B_k to accuracy `prec` is enough to reconstruct it exactly, a `t_FRAC` is stored; otherwise a `t_REAL` at the requested accuracy. No more than n Bernoulli numbers will ever be stored (by `bernfrac` or `bernreal`), unless a subsequent call to `mpbern` increases the cache. If `prec` is 0, the B_k are computed exactly.

The following functions use cached data if `prec` is smaller than the precision of the cached value; otherwise the newly computed data replaces the old cache.

GEN `mppi(long prec)` returns π at precision `prec`.

GEN `Pi2n(long n, long prec)` returns $2^n\pi$ at precision `prec`.

GEN `PiI2(long n, long prec)` returns the complex number $2\pi i$ at precision `prec`.

GEN `PiI2n(long n, long prec)` returns the complex number $2^n\pi i$ at precision `prec`.

GEN `mpeuler(long prec)` returns Euler-Mascheroni's constant at precision `prec`.

GEN `mpeuler(long prec)` returns Catalan's number at precision `prec`.

GEN `mplog2(long prec)` returns $\log 2$ at precision `prec`.

GEN `bernreal(long i, long prec)` returns the Bernoulli number B_i as a `t_REAL` at precision `prec`. If `mpbern(n, p)` was called previously with $n \geq i$ and $p \geq \text{prec}$, then the cached value is (converted to a `t_REAL` of accuracy `prec` then) returned. Otherwise, the missing value is computed. In the latter case, if $n \geq i$, the cached table is updated.

GEN `bernfrac(long i)` returns the Bernoulli number B_i as a rational number (`t_FRAC` or `t_INT`). If a cached table includes B_i as a rational number, the latter is returned. Otherwise, the missing value is computed. In the latter case, the cached Bernoulli table may be updated.

10.10 Permutations .

Permutations are represented in two different ways

- (`perm`) a `t_VECSMALL` p representing the bijection $i \mapsto p[i]$; unless mentioned otherwise, this is the form used in the functions below for both input and output,

- (`cyc`) a `t_VEC` of `t_VECSMALLs` representing a product of disjoint cycles.

GEN `identity_perm(long n)` return the identity permutation on n symbols.

GEN `cyclic_perm(long n, long d)` return the cyclic permutation mapping i to $i + d \pmod{n}$ in S_n . Assume that $d \leq n$.

GEN `perm_mul(GEN s, GEN t)` multiply s and t (composition $s \circ t$)

GEN `perm_conj(GEN s, GEN t)` return sts^{-1} .

int `perm_commute(GEN p, GEN q)` return 1 if p and q commute, 0 otherwise.

GEN `perm_inv(GEN p)` returns the inverse of p .

GEN `perm_pow(GEN p, long n)` returns p^n

GEN `cyc_pow_perm(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `perm`).

GEN `cyc_pow(GEN p, long n)` the permutation p is given as a product of disjoint cycles (`cyc`); return p^n (as a `cyc`).

GEN `perm_cycles(GEN p)` return the cyclic decomposition of p .

long `perm_order(GEN p)` returns the order of the permutation p (as the lcm of its cycle lengths).

`long perm_sign(GEN p)` returns the sign of the permutation p .

`GEN vecperm_orbits(GEN p, long n)` the permutation $p \in S_n$ being given as a product of disjoint cycles, return the orbits of the subgroup generated by p on $\{1, 2, \dots, n\}$.

`GEN Z_to_perm(long n, GEN x)` as `numtoperm`, returning a `t_VECSMALL`.

`GEN perm_to_Z(GEN v)` as `permtonum` for a `t_VECSMALL` input.

10.11 Small groups.

The small (finite) groups facility is meant to deal with subgroups of Galois groups obtained by `galoisinit` and thus is currently limited to weakly super-solvable groups.

A group grp of order n is represented by its regular representation (for an arbitrary ordering of its element) in S_n . A subgroup of such group is represented by the restriction of the representation to the subgroup. A *small group* can be either a group or a subgroup. Thus it is embedded in some S_n , where n is the multiple of the order. Such an n is called the *domain* of the small group. The domain of a trivial subgroup cannot be derived from the subgroup data, so some functions require the subgroup domain as argument.

The small group grp is represented by a `t_VEC` with two components:

`grp[1]` is a generating subset $[s_1, \dots, s_g]$ of grp expressed as a vector of permutations of length n .

`grp[2]` contains the relative orders $[o_1, \dots, o_g]$ of the generators $grp[1]$.

See `galoisinit` for the technical details.

`GEN checkgroup(GEN gal, GEN *elts)` check whether gal is a small group or a Galois group. Returns the underlying small group and set $elts$ to the list of elements or to `NULL` if it is not known.

`GEN checkgroupeelts(GEN gal)` check whether gal is a small group or a Galois group, or a vector of permutations listing the group elements. Returns the list of group elements as permutations.

`GEN galois_group(GEN gal)` return the underlying small group of the Galois group gal .

`GEN cyclicgroup(GEN g, long s)` return the cyclic group with generator g of order s .

`GEN trivialgroup(void)` return the trivial group.

`GEN dicyclicgroup(GEN g1, GEN g2, long s1, long s2)` returns the group with generators $g1, g2$ with respecting relative orders $s1, s2$.

`GEN abelian_group(GEN v)` let v be a `t_VECSMALL` seen as the SNF of a small abelian group, return its regular representation.

`long group_domain(GEN grp)` returns the domain of the *non-trivial* small group grp . Return an error if grp is trivial.

`GEN group_elts(GEN grp, long n)` returns the list of elements of the small group grp of domain n as permutations.

`GEN group_set(GEN grp, long n)` returns a $F2v$ b such that $b[i]$ is set if and only if the small group grp of domain n contains a permutation sending 1 to i .

`GEN groupeelts_set(GEN elts, long n)`, where $elts$ is the list of elements of a small group of domain n , returns a $F2v$ b such that $b[i]$ is set if and only if the small group contains a permutation sending 1 to i .

GEN `groupelts_conjclasses`(GEN `elts`, long `*pn`), where `elts` is the list of elements of a small group (sorted with respect to `vecsmall_lexcmp`), return a `t_VECSMALL` `conj` of the same length such that `conj[i]` is the index in $\{1, \dots, n\}$ of the conjugacy class of `elts[i]` for some unspecified but deterministic ordering of the classes, where n is the number of conjugacy classes. If `pn` is non NULL, `*pn` is set to n .

GEN `conjclasses_repr`(GEN `conj`, long `nb`), where `conj` and `nb` are as returned by the call `groupelts_conjclasses(elts)`, return `t_VECSMALL` of length `nb` which gives the indices in `elts` of a representative of each conjugacy class.

GEN `group_to_cc`(GEN `G`), where G is a small group or a Galois group, returns a `cc` (conjugacy classes) structure `[elts,conj,rep,flag]`, as obtained by `alggroupcenter`, where `conj` is `groupelts_conjclasses(elts)` and `rep` is the attached `conjclasses_repr`. `flag` is 1 if the permutation representation is transitive (in which case an element g of G is characterized by `g[1]`), and 0 otherwise. Shallow function.

long `group_order`(GEN `grp`) returns the order of the small group `grp` (which is the product of the relative orders).

long `group_isabelian`(GEN `grp`) returns 1 if the small group `grp` is Abelian, else 0.

GEN `group_abelianHNF`(GEN `grp`, GEN `elts`) if `grp` is not Abelian, returns NULL, else returns the HNF matrix of `grp` with respect to the generating family `grp[1]`. If `elts` is no NULL, it must be the list of elements of `grp`.

GEN `group_abelianSNF`(GEN `grp`, GEN `elts`) if `grp` is not Abelian, returns NULL, else returns its cyclic decomposition. If `elts` is no NULL, it must be the list of elements of `grp`.

long `group_subgroup_isnormal`(GEN `G`, GEN `H`), H being a subgroup of the small group G , returns 1 if H is normal in G , else 0.

long `group_isA4S4`(GEN `grp`) returns 1 if the small group `grp` is isomorphic to A_4 , 2 if it is isomorphic to S_4 and 0 else. This is mainly to deal with the idiosyncrasy of the format.

GEN `group_leftcoset`(GEN `G`, GEN `g`) where G is a small group and g a permutation of the same domain, the left coset gG as a vector of permutations.

GEN `group_rightcoset`(GEN `G`, GEN `g`) where G is a small group and g a permutation of the same domain, the right coset Gg as a vector of permutations.

long `group_perm_normalize`(GEN `G`, GEN `g`) where G is a small group and g a permutation of the same domain, return 1 if $gGg^{-1} = G$, else 0.

GEN `group_quotient`(GEN `G`, GEN `H`), where G is a small group and H is a subgroup of G , returns the quotient map $G \rightarrow G/H$ as an abstract data structure.

GEN `quotient_perm`(GEN `C`, GEN `g`) where C is the quotient map $G \rightarrow G/H$ for some subgroup H of G and g an element of G , return the image of g by C (i.e. the coset gH).

GEN `quotient_group`(GEN `C`, GEN `G`) where C is the quotient map $G \rightarrow G/H$ for some normal subgroup H of G , return the quotient group G/H as a small group.

GEN `quotient_subgroup_lift`(GEN `C`, GEN `H`, GEN `S`) where C is the quotient map $G \rightarrow G/H$ for some group G normalizing H and S is a subgroup of G/H , return the inverse image of S by C .

GEN `group_subgroups`(GEN `grp`) returns the list of subgroups of the small group `grp` as a `t_VEC`.

GEN `subgroups_tableset(GEN S, long n)` where S is a vector of subgroups of domain n , returns a table which matches the set of elements of the subgroups against the index of the subgroups.

long `tableset_find_index(GEN tbl, GEN set)` searches the set `set` in the table `tbl` and returns its attached index, or 0 if not found.

GEN `groupeelts_abelian_group(GEN elts)` where $elts$ is the list of elements of an *Abelian* small group, returns the corresponding small group.

long `groupeelts_exponent(GEN elts)` where $elts$ is the list of elements of a small group, returns the exponent the group (the LCM of the order of the elements of the group).

GEN `groupeelts_center(GEN elts)` where $elts$ is the list of elements of a small group, returns the list of elements of the center of the group.

GEN `group_export(GEN grp, long format)` convert a small group to another format, as a `t_STR` describing the group for the given syntax, see `galoisexport`.

GEN `group_export_GAP(GEN G)` export a small group to GAP format.

GEN `group_export_MAGMA(GEN G)` export a small group to MAGMA format.

long `group_ident(GEN grp, GEN elts)` returns the index of the small group grp in the GAP4 Small Group library, see `galoisidentify`. If $elts$ is not NULL, it must be the list of elements of grp .

long `group_ident_trans(GEN grp, GEN elts)` returns the index of the regular representation of the small group grp in the GAP4 Transitive Group library, see `polgalois`. If $elts$ is no NULL, it must be the list of elements of grp .

Chapter 11:

Standard data structures

11.1 Character strings.

11.1.1 Functions returning a char *.

`char* pari_strdup(const char *s)` returns a malloc'ed copy of *s* (uses `pari_malloc`).

`char* pari_strndup(const char *s, long n)` returns a malloc'ed copy of at most *n* chars from *s* (uses `pari_malloc`). If *s* is longer than *n*, only *n* characters are copied and a terminal null byte is added.

`char* stack_strdup(const char *s)` returns a copy of *s*, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_strcat(const char *s, const char *t)` returns the concatenation of *s* and *t*, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_sprintf(const char *fmt, ...)` runs `pari_sprintf` on the given arguments, returning a string allocated on the PARI stack.

`char* uordinal(ulong x)` return the ordinal number attached to *x* (i.e. 1st, 2nd, etc.) as a `stack_malloc`'ed string.

`char* itostr(GEN x)` writes the `t_INT` *x* to a `stack_malloc`'ed string.

`char* GENTostr(GEN x)`, using the current default output format (`GP_DATA->fmt`, which contains the output style and the number of significant digits to print), converts *x* to a malloc'ed string. Simple variant of `pari_sprintf`.

`char* GENTostr_raw(GEN x)` as `GENTostr` with the following differences: 1) the output format is `f_RAW`; 2) the result is allocated on the stack and *must not* be freed.

`char* GENTostr_unquoted(GEN x)` as `GENTostr_raw` with the following additional difference: a `t_STR` *x* is printed without enclosing quotes (to be used by `print`).

`char* GENToTeXstr(GEN x)`, as `GENTostr`, except that `f_TEX` overrides the output format from `GP_DATA->fmt`.

`char* RgV_to_str(GEN g, long flag)` *g* being a vector of GENs, returns a malloc'ed string, the concatenation of the `GENTostr` applied to its elements, except that `t_STR` are printed without enclosing quotes. `flag` determines the output format: `f_RAW`, `f_PRETTYMAT` or `f_TEX`.

11.1.2 Functions returning a `t_STR`.

GEN `strtoGENstr(const char *s)` returns a `t_STR` with content `s`.

GEN `strntoGENstr(const char *s, long n)` returns a `t_STR` containing the first `n` characters of `s`.

GEN `chartoGENstr(char c)` returns a `t_STR` containing the character `c`.

GEN `GENtoGENstr(GEN x)` returns a `t_STR` containing the printed form of `x` (in `raw` format). This is often easier to use than `GENtostr` (which returns a malloc'ed `char*`) since there is no need to free the string after use.

GEN `GENtoGENstr_nospace(GEN x)` as `GENtoGENstr`, removing all spaces from the output.

GEN `Str(GEN g)` as `RgV_to_str` with output format `f_RAW`, but returns a `t_STR`, not a malloc'ed string.

GEN `Strtex(GEN g)` as `RgV_to_str` with output format `f_TEX`, but returns a `t_STR`, not a malloc'ed string.

GEN `Strexpend(GEN g)` as `RgV_to_str` with output format `f_RAW`, performing tilde and environment expansion on the result. Returns a `t_STR`, not a malloc'ed string.

GEN `gsprintf(const char *fmt, ...)` equivalent to `pari_sprintf(fmt, ...)`, followed by `strtoGENstr`. Returns a `t_STR`, not a malloc'ed string.

GEN `gvsprintf(const char *fmt, va_list ap)` variadic version of `gsprintf`

11.1.3 Dynamic strings.

A `pari_str` is a dynamic string which grows dynamically as needed. This structure contains private data and two public members `char *string`, which is the string itself and `use_stack` which tells whether the string lives

- on the PARI stack (value 1), meaning that it will be destroyed by any manipulation of the stack, e.g. a `gerepile` call or resetting `avma`;
- in malloc'ed memory (value 0), in which case it is impervious to stack manipulation but will need to be explicitly freed by the user after use, via `pari_free(s.string)`.

`void str_init(pari_str *S, int use_stack)` initializes a dynamic string; if `use_stack` is 0, then the string is malloc'ed, else it lives on the PARI stack.

`void str_printf(pari_str *S, const char *fmt, ...)` write to the end of `S` the remaining arguments according to PARI format `fmt`.

`void str_putc(pari_str *S, char c)` write the character `c` to the end of `S`.

`void str_puts(pari_str *S, const char *s)` write the string `s` to the end of `S`.

11.2 Output.

11.2.1 Output contexts.

An output context, of type `PariOUT`, is a `struct` that models a stream and contains the following function pointers:

```
void (*putch)(char);           /* fputc()-alike */
void (*puts)(const char*);    /* fputs()-alike */
void (*flush)(void);          /* fflush()-alike */
```

The methods `putch` and `puts` are used to print a character or a string respectively. The method `flush` is called to finalize a messages.

The generic functions `pari_putc`, `pari_puts`, `pari_flush` and `pari_printf` print according to a *default output context*, which should be sufficient for most purposes. Lower level functions are available, which take an explicit output context as first argument:

`void out_putc(PariOUT *out, char c)` essentially equivalent to `out->putc(c)`. In addition, registers whether the last character printed was a `\n`.

`void out_puts(PariOUT *out, const char *s)` essentially equivalent to `out->puts(s)`. In addition, registers whether the last character printed was a `\n`.

`void out_printf(PariOUT *out, const char *fmt, ...)`

`void out_vprintf(PariOUT *out, const char *fmt, va_list ap)`

N.B. The function `out_flush` does not exist since it would be identical to `out->flush()`

`int pari_last_was_newline(void)` returns a non-zero value if the last character printed via `out_putc` or `out_puts` was `\n`, and 0 otherwise.

`void pari_set_last_newline(int last)` sets the boolean value to be returned by the function `pari_last_was_newline` to *last*.

11.2.2 Default output context. They are defined by the global variables `pariOut` and `pariErr` for normal outputs and warnings/errors, and you probably do not want to change them. If you *do* change them, diverting output in non-trivial ways, this probably means that you are rewriting `gp`. For completeness, we document in this section what the default output contexts do.

pariOut. writes output to the `FILE*` `pari_outfile`, initialized to `stdout`. The low-level methods are actually the standard `putc` / `fputs`, plus some magic to handle a log file if one is open.

pariErr. prints to the `FILE*` `pari_errfile`, initialized to `stderr`. The low-level methods are as above.

You can stick with the default `pariOut` output context and change PARI's standard output, redirecting `pari_outfile` to another file, using

`void switchout(const char *name)` where `name` is a character string giving the name of the file you want to write to; the output is *appended* at the end of the file. To close the file and revert to outputting to `stdout`, call `switchout(NULL)`.

11.2.3 PARI colors. In this section we describe the low-level functions used to implement GP's color scheme, attached to the `colors` default. The following symbolic names are attached to gp's output strings:

- `c_ERR` an error message
- `c_HIST` a history number (as in `%1 = ...`)
- `c_PROMPT` a prompt
- `c_INPUT` an input line (minus the prompt part)
- `c_OUTPUT` an output
- `c_HELP` a help message
- `c_TIME` a timer
- `c_NONE` everything else

If the `colors` default is set to a non-empty value, before gp outputs a string, it first outputs an ANSI colors escape sequence — understood by most terminals —, according to the `colors` specifications. As long as this is in effect, the following strings are rendered in color, possibly in bold or underlined.

`void term_color(long c)` prints (as if using `pari_puts`) the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, revert to default printing style.

`void out_term_color(PariOUT *out, long c)` as `term_color`, using output context `out`.

`char* term_get_color(char *s, long c)` returns as a character string the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, the value used to revert to default printing style is returned. The argument `s` is either `NULL` (string allocated on the PARI stack), or preallocated storage (in which case, it must be able to hold at least 16 chars, including the final `\0`).

11.2.4 Obsolete output functions.

These variants of `void output(GEN x)`, which prints `x`, followed by a newline and a buffer flush are complicated to use and less flexible than what we saw above, or than the `pari_printf` variants. They are provided for backward compatibility and are scheduled to disappear.

`void brute(GEN x, char format, long dec)`

`void matbrute(GEN x, char format, long dec)`

`void texe(GEN x, char format, long dec)`

11.3 Files.

The following routines are trivial wrappers around system functions (possibly around one of several functions depending on availability). They are usually integrated within PARI's diagnostics system, printing messages if `DEBUGFILES` is high enough.

`int pari_is_dir(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int pari_is_file(const char *name)` returns 1 if `name` points to a file, 0 otherwise.

`int file_is_binary(FILE *f)` returns 1 if the file `f` is a binary file (in the `writebin` sense), 0 otherwise.

`void pari_unlink(const char *s)` deletes the file named `s`. Warn if the operation fails.

`void pari_fread_chars(void *b, size_t n, FILE *f)` read `n` chars from stream `f`, storing the result in pre-allocated buffer `b` (assumed to be large enough).

`char* path_expand(const char *s)` perform tilde and environment expansion on `s`. Returns a malloc'ed buffer.

`void strftime_expand(const char *s, char *buf, long max)` perform time expansion on `s`, storing the result (at most `max` chars) in buffer `buf`. Trivial wrapper around

```
time_t t = time(NULL);
strftime(buf, max, s, localtime(&t));
```

`char* pari_get_homedir(const char *user)` expands `~user` constructs, returning the home directory of user `user`, or `NULL` if it could not be determined (in particular if the operating system has no such concept). The return value may point to static area and may be overwritten by subsequent system calls: use immediately or `strdup` it.

`int pari_stdin_isatty(void)` returns 1 if our standard input `stdin` is attached to a terminal. Trivial wrapper around `isatty`.

11.3.1 pariFILE.

PARI maintains a linked list of open files, to reclaim resources (file descriptors) on error or interrupts. The corresponding data structure is a `pariFILE`, which is a wrapper around a standard `FILE*`, containing further the file name, its type (regular file, pipe, input or output file, etc.). The following functions create and manipulate this structure; they are integrated within PARI's diagnostics system, printing messages if `DEBUGFILES` is high enough.

`pariFILE* pari_fopen(const char *s, const char *mode)` wrapper around `fopen(s, mode)`, return `NULL` on failure.

`pariFILE* pari_fopen_or_fail(const char *s, const char *mode)` simple wrapper around `fopen(s, mode)`; error on failure.

`pariFILE* pari_fopengz(const char *s)` opens the file whose name is `s`, and associates a (read-only) `pariFILE` with it. If `s` is a compressed file (`.gz` suffix), it is uncompressed on the fly. If `s` cannot be opened, also try to open `s.gz`. Returns `NULL` on failure.

`void pari_fclose(pariFILE *f)` closes the underlying file descriptor and deletes the `pariFILE` struct.

`pariFILE* pari_safefopen(const char *s, const char *mode)` creates a *new* file `s` (a priori for writing) with 600 permissions. Error if the file already exists. To avoid symlink attacks, a symbolic link exists, regardless of where it points to.

11.3.2 Temporary files.

PARI has its own idea of the system temp directory derived from an environment variable (\$GPTMPDIR, else \$TMPDIR), or the first writable directory among /tmp, /var/tmp and ..

`char* pari_unique_dir(const char *s)` creates a “unique directory” and return its name built from the string *s*, the user id and process pid (on Unix systems). This directory is itself located in the temp directory mentioned above. The name returned is `malloc`'ed.

`char* pari_unique_filename(const char *s)` creates a *new* empty file in the temp directory, whose name contains the id-string *s* (truncated to its first 8 chars), followed by a system-dependent suffix (incorporating the ids of both the user and the running process, for instance). The function returns the tempfile name and creates an empty file with that name. The name returned is `malloc`'ed.

`char* pari_unique_filename_suffix(const char *s, const char *suf)` analogous to above `pari_unique_filename`, creating a (previously non-existent) tempfile whose name ends with suffix *suf*.

11.4 Errors.

This section documents the various error classes, and the corresponding arguments to `pari_err`. The general syntax is

```
void pari_err(numerr, ...)
```

In the sequel, we mostly use sequences of arguments of the form

```
const char *s
const char *fmt, ...
```

where *fmt* is a PARI format, producing a string *s* from the remaining arguments. Since providing the correct arguments to `pari_err` is quite error-prone, we also provide specialized routines `pari_err_ERRORCLASS(...)` instead of `pari_err(e_ERRORCLASS, ...)` so that the C compiler can check their arguments.

We now inspect the list of valid keywords (error classes) for `numerr`, and the corresponding required arguments.

11.4.1 Internal errors, “system” errors.

11.4.1.1 e_ARCH. A requested feature *s* is not available on this architecture or operating system.

```
pari_err(e_ARCH)
```

prints the error message: `sorry, 's' not available on this system.`

11.4.1.2 e_BUG. A bug in the PARI library, in function *s*.

```
pari_err(e_BUG, const char *s)
pari_err_BUG(const char *s)
```

prints the error message: `Bug in s, please report.`

11.4.1.3 e_FILE. Error while trying to open a file.

```
pari_err(e_FILE, const char *what, const char *name)
pari_err_FILE(const char *what, const char *name)
```

prints the error message: error opening *what*: '*name*'.

11.4.1.4 e_FILEDESC. Error while handling a file descriptor.

```
pari_err(e_FILEDESC, const char *where, long n)
pari_err_FILEDESC(const char *where, long n)
```

prints the error message: invalid file descriptor in *where*: '*name*'.

11.4.1.5 e_IMPL. A requested feature *s* is not implemented.

```
pari_err(e_IMPL, const char *s)
pari_err_IMPL(const char *s)
```

prints the error message: sorry, *s* is not yet implemented.

11.4.1.6 e_PACKAGE. Missing optional package *s*.

```
pari_err(e_PACKAGE, const char *s)
pari_err_PACKAGE(const char *s)
```

prints the error message: package *s* is required, please install it

11.4.2 Syntax errors, type errors.

11.4.2.1 e_DIM. arguments submitted to function *s* have inconsistent dimensions. E.g., when solving a linear system, or trying to compute the determinant of a non-square matrix.

```
pari_err(e_DIM, const char *s)
pari_err_DIM(const char *s)
```

prints the error message: inconsistent dimensions in *s*.

11.4.2.2 e_FLAG. A flag argument is out of bounds in function *s*.

```
pari_err(e_FLAG, const char *s)
pari_err_FLAG(const char *s)
```

prints the error message: invalid flag in *s*.

11.4.2.3 e_NOTFUNC. Generated by the PARI evaluator; tried to use a GEN which is not a t_CLOSURE in a function call syntax (as in `f = 1; f(2);`).

```
pari_err(e_NOTFUNC, GEN fun)
```

prints the error message: not a function in a function call.

11.4.2.4 e_OP. Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in $0(2) + 0(3)$: it is valid to add two t_PADICs, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation.

```
pari_err(e_OP, const char *op, GEN x, GEN y)
pari_err_OP(const char *op, GEN x, GEN y)
```

As e.TYPE2, replacing forbidden by inconsistent.

11.4.2.5 e_PRIORITY. object o in function s contains variables whose priority is incompatible with the expected operation. E.g. `Pol([x,1], 'y)`: this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable.

```
pari_err(e_PRIORITY, const char *s, GEN o, const char *op, long v)
pari_err_PRIORITY(const char *s, GEN o, const char *op, long v)
```

prints the error message: `incorrect priority in s, variable v_o op v , were v_o is gvar(o).`

11.4.2.6 e_SYNTAX. Syntax error, generated by the PARI parser.

```
pari_err(e_SYNTAX, const char *msg, const char *e, const char *entry)
```

where `msg` is a complete error message, and `e` and `entry` point into the *same* character string, which is the input that was incorrectly parsed: `e` points to the character where the parser failed, and `entry` \leq `e` points somewhat before.

Prints the error message: `msg`, followed by a colon, then a part of the input character string (in general `entry` itself, but an initial segment may be truncated if `e - entry` is large); a caret points at `e`, indicating where the error took place.

11.4.2.7 e_TYPE. An argument x of function s had an unexpected type. (As in `factor("blah")`.)

```
pari_err(e_TYPE, const char *s, GEN x)
pari_err_TYPE(const char *s, GEN x)
```

prints the error message: `incorrect type in s (t_x)`, where t_x is the type of x .

11.4.2.8 e_TYPE2. Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in `Mod(1,2) + Pi`.)

```
pari_err(e_TYPE2, const char *op, GEN x, GEN y)
pari_err_TYPE2(const char *op, GEN x, GEN y)
```

prints the error message: `forbidden $s t_x$ op t_y` , where t_z denotes the type of z . Here, s denotes the spelled out name of the operator $op \in \{+, *, /, \%, =\}$, e.g. *addition* for "+" or *assignment* for "=". If op is not in the above operator, list, it is taken to be the already spelled out name of a function, e.g. "gcd", and the error message becomes `forbidden $op t_x, t_y$` .

11.4.2.9 e_VAR. polynomials x and y submitted to function s have inconsistent variables. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfini(x^2+1)`.

```
pari_err(e_VAR, const char *s, GEN x, GEN y)
pari_err_VAR(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent variables in s $X \neq Y$` , where X and Y are the names of the variables of x and y , respectively.

11.4.3 Overflows.

11.4.3.1 e_COMPONENT. Trying to access an inexistent component of a vector/matrix/list: the index is less than 1 or greater than the allowed length.

```
pari_err(e_COMPONENT, const char *f, const char *op, GEN lim, GEN x)
pari_err_COMPONENT(const char *f, const char *op, GEN lim, GEN x)
```

prints the error message: `non-existent component in f: index op lim` . Special case: if f is the empty string (no meaningful public function name can be used), we ignore it and print the message: `non-existent component: index op lim` .

11.4.3.2 e_DOMAIN. An argument x is not in the function's domain (as in `moebius(0)` or `zeta(1)`).

```
pari_err(e_DOMAIN, char *f, char *v, char *op, GEN lim, GEN x)
pari_err_DOMAIN(char *f, char *v, char *op, GEN lim, GEN x)
```

prints the error message: `domain error in f: v op lim`. Special case: if `op` is the empty string, we ignore `lim` and print the error message: `domain error in f: v out of range`.

11.4.3.3 e_MAXPRIME. A function using the precomputed list of prime numbers ran out of primes.

```
pari_err(e_MAXPRIME, ulong c)
pari_err_MAXPRIME(ulong c)
```

prints the error message: `not enough precomputed primes, need primelimit ~c` if c is non-zero. And simply `not enough precomputed primes` otherwise.

11.4.3.4 e_MEM. A call to `pari_malloc` or `pari_realloc` failed.

```
pari_err(e_MEM)
```

prints the error message: `not enough memory`.

11.4.3.5 e_OVERFLOW. An object in function s becomes too large to be represented within PARI's hardcoded limits. (As in `2^2^2^10` or `exp(1e100)`, which overflow in `lg` and `expo`.)

```
pari_err(e_OVERFLOW, const char *s)
pari_err_OVERFLOW(const char *s)
```

prints the error message: `overflow in s`.

11.4.3.6 e_PREC. Function s fails because input accuracy is too low. (As in `floor(1e100)` at default accuracy.)

```
pari_err(e_PREC, const char *s)
pari_err_PREC(const char *s)
```

prints the error message: `precision too low in s`.

11.4.3.7 e_STACK. The PARI stack overflows.

```
pari_err(e_STACK)
```

prints the error message: `the PARI stack overflows !` as well as some statistics concerning stack usage.

11.4.4 Errors triggered intentionally.

11.4.4.1 e_ALARM. A timeout, generated by the `alarm` function.

```
pari_err(e_ALARM, const char *fmt, ...)
```

prints the error message: `s`.

11.4.4.2 e_USER. A user error, as triggered by `error(g1, ..., gn)` in GP.

```
pari_err(e_USER, GEN g)
```

prints the error message: `user error:`, then the entries of the vector g .

11.4.5 Mathematical errors.

11.4.5.1 e_CONSTPOL. An argument of function s is a constant polynomial, which does not make sense. (As in `galoisinit(Pol(1))`.)

```
pari_err(e_CONSTPOL, const char *s)
pari_err_CONSTPOL(const char *s)
```

prints the error message: `constant polynomial in s`.

11.4.5.2 e_COPRIME. Function s expected two coprime arguments, and did receive x, y which were not.

```
pari_err(e_COPRIME, const char *s, GEN x, GEN y)
pari_err_COPRIME(const char *s, GEN x, GEN y)
```

prints the error message: `elements not coprime in s: x,y`.

11.4.5.3 e_INV. Tried to invert a non-invertible object x .

```
pari_err(e_INV, const char *s, GEN x)
pari_err_INV(const char *s, GEN x)
```

prints the error message: `impossible inverse in s: x`. If $x = \text{Mod}(a, b)$ is a `t_INTMOD` and a is not 0 mod b , this allows to factor the modulus, as $\text{gcd}(a, b)$ is a non-trivial divisor of b .

11.4.5.4 e_IRREDPOL. Function s expected an irreducible polynomial, and did not receive one. (As in `nfinit(x^2-1)`.)

```
pari_err(e_IRREDPOL, const char *s, GEN x)
pari_err_IRREDPOL(const char *s, GEN x)
```

prints the error message: `not an irreducible polynomial in s: x`.

11.4.5.5 e_MISC. Generic uncategorized error.

```
pari_err(e_MISC, const char *fmt, ...)
```

prints the error message: `s`.

11.4.5.6 e_MODULUS. moduli x and y submitted to function s are inconsistent. E.g., considering the algebraic number $\text{Mod}(t, t^2+1)$ in `nfinit(t^3-2)`.

```
pari_err(e_MODULUS, const char *s, GEN x, GEN y)
pari_err_MODULUS(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent moduli in s, then the moduli`.

11.4.5.7 e_PRIME. Function s expected a prime number, and did receive p , which was not. (As in `idealprimedec(nf, 4)`.)

```
pari_err(e_PRIME, const char *s, GEN x)
pari_err_PRIME(const char *s, GEN x)
```

prints the error message: `not a prime in s: x`.

11.4.5.8 e_ROOTS0. An argument of function s is a zero polynomial, and we need to consider its roots. (As in `polroots(0)`.)

```
pari_err(e_ROOTS0, const char *s)
pari_err_ROOTS0(const char *s)
```

prints the error message: `zero polynomial in s`.

11.4.5.9 e_SQR TN. Tried to compute an n -th root of x , which does not exist, in function s . (As in `sqrt(Mod(-1,3))`.)

```
pari_err(e_SQR TN, GEN x)
pari_err_SQR TN(GEN x)
```

prints the error message: `not an n-th power residue in s: x`.

11.4.6 Miscellaneous functions.

`long name_numerr(const char *s)` return the error number corresponding to an error name. E.g. `name_numerr("e_DIM")` returns `e_DIM`.

`const char* numerr_name(long errnum)` returns the error name corresponding to an error number. E.g. `name_numerr(e_DIM)` returns `"e_DIM"`.

`char* pari_err2str(GEN err)` returns the error message that would be printed on `t_ERROR err`. The name is allocated on the PARI stack and must not be freed.

11.5 Hashtables.

A **hashtable**, or associative array, is a set of pairs (k, v) of keys and values. PARI implements general extensible hashtables for fast data retrieval: when creating a table, we may either choose to use the PARI stack, or `malloc` so as to be stack-independent. A hashtable is implemented as a table of linked lists, each list containing all entries sharing the same hash value. The table length is a prime number, which roughly doubles as the table overflows by gaining new entries; both the current number of entries and the threshold before the table grows are stored in the table. Finally the table remembers the functions used to hash the entries's keys and to test for equality two entries hashed to the same value.

An entry, or **hashentry**, contains

- a key/value pair (k, v) , both of type `void*` for maximal flexibility,
- the hash value of the key, for the table hash function. This hash is mapped to a table index (by reduction modulo the table length), but it contains more information, and is used to bypass costly general equality tests if possible,
- a link pointer to the next entry sharing the same table cell.

```
typedef struct {
    void *key, *val;
    ulong hash; /* hash(key) */
    struct hashentry *next;
} hashentry;

typedef struct {
    ulong len; /* table length */
```

```

    hashentry **table; /* the table */
    ulong nb, maxnb; /* number of entries stored and max nb before enlarging */
    ulong pindex; /* prime index */
    ulong (*hash) (void *k); /* hash function */
    int (*eq) (void *k1, void *k2); /* equality test */
    int use_stack; /* use the PARI stack, resp. malloc */
} hashtable;

```

```

hashtable* hash_create(size, hash, eq, use_stack)
    ulong size;
    ulong (*hash)(void*);
    int (*eq)(void*,void*);
    int use_stack;

```

creates a hashtable with enough room to contain `size` entries. The functions `hash` and `eq` compute the hash value of keys and test keys for equality, respectively. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`hashtable* hash_create_ulong(ulong size, long stack)` special case when the keys are ulongs with ordinary equality test.

`hashtable* hash_create_str(ulong size, long stack)` special case when the keys are character strings with string equality test (and `hash_str` hash function).

`void hash_init_GEN(hashtable *h, ulong size, int (*eq)(GEN, GEN), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `GEN`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`. The hash used is `hash_GEN`.

`void hash_insert(hashtable *h, void *k, void *v)` inserts (k, v) in hashtable `h`. No copy is made: `k` and `v` themselves are stored. The implementation does not prevent one to insert two entries with equal keys `k`, but which of the two is affected by later commands is undefined.

`void hash_insert2(hashtable *h, void *k, void *v, ulong hash)` as `hash_insert`, assuming `h->hash(k)` is `hash`.

`void hash_insert_long(hashtable *h, void *k, long v)` as `hash_insert` but `v` is a long.

`hashentry* hash_search(hashtable *h, void *k)` look for an entry with key `k` in `h`. Return it if it one exists, and `NULL` if not.

`hashentry* hash_search2(hashtable *h, void *k, ulong hash)` as `hash_search` assuming `h->hash(k)` is `hash`.

`int hash_haskey_long(hashtable *h, void *k, long *v)` returns 1 if the key `k` belongs to the hash and set `v` to its value, otherwise returns 0.

`hashentry * hash_select(hashtable *h, void *k, void *E, int (*select)(void *, hashentry *))` variant of `hash_search`, useful when entries with identical keys are inserted: among the entries attached to key `k`, return one satisfying the selection criterion (such that `select(E,e)` is non-zero), or `NULL` if none exist.

`hashentry* hash_remove(hashtable *h, void *k)` deletes an entry (k, v) with key `k` from `h` and return it. (Return `NULL` if none was found.) Only the linking structures are freed, memory attached to `k` and `v` is not reclaimed.

`hashentry* hash_remove_select(hashtable *h, void *k, void *E, int(*select)(void*, hashentry *))` a variant of `hash_remove`, useful when entries with identical keys are inserted: among the entries attached to key k , return one satisfying the selection criterion (such that `select(E,e)` is non-zero) and delete it, or `NULL` if none exist. Only the linking structures are freed, memory attached to k and v is not reclaimed.

`GEN hash_keys(hashtable *h)` return in a `t_VECSMALL` the keys stored in hashtable h .

`GEN hash_values(hashtable *h)` return in a `t_VECSMALL` the values stored in hashtable h .

`void hash_destroy(hashtable *h)` deletes the hashtable, by removing all entries.

`void hash_dbg(hashtable *h)` print statistics for hashtable h , allows to evaluate the attached hash function performance on actual data.

Some interesting hash functions are available:

`ulong hash_str(const char *s)`

`ulong hash_str2(const char *s)` is the historical PARI string hashing function and seems to be generally inferior to `hash_str`.

`ulong hash_GEN(GEN x)`

11.6 Dynamic arrays.

A **dynamic array** is a generic way to manage stacks of data that need to grow dynamically. It allocates memory using `pari_malloc`, and is independent of the PARI stack; it even works before the `pari_init` call.

11.6.1 Initialization.

To create a stack of objects of type `foo`, we proceed as follows:

```
foo *t_foo;
pari_stack s_foo;
pari_stack_init(&s_foo, sizeof(*t_foo), (void**)&t_foo);
```

Think of `s_foo` as the controlling interface, and `t_foo` as the (dynamic) array tied to it. The value of `t_foo` may be changed as you add more elements.

11.6.2 Adding elements. The following function pushes an element on the stack.

```
/* access globals t_foo and s_foo */
void push_foo(foo x)
{
    long n = pari_stack_new(&s_foo);
    t_foo[n] = x;
}
```

11.6.3 Accessing elements.

Elements are accessed naturally through the `t_foo` pointer. For example this function swaps two elements:

```
void swapfoo(long a, long b)
{
    foo x;
    if (a > s_foo.n || b > s_foo.n) pari_err_BUG("swapfoo");
    x          = t_foo[a];
    t_foo[a] = t_foo[b];
    t_foo[b] = x;
}
```

11.6.4 Stack of stacks. Changing the address of `t_foo` is not supported in general. In particular `realloc()`'ed array of stacks and stack of stacks are not supported.

11.6.5 Public interface. Let `s` be a `pari_stack` and `data` the data linked to it. The following public fields are defined:

- `s.alloc` is the number of elements allocated for `data`.
- `s.n` is the number of elements in the stack and `data[s.n-1]` is the topmost element of the stack. `s.n` can be changed as long as $0 \leq s.n \leq s.alloc$ holds.

`void pari_stack_init(pari_stack *s, size_t size, void **data)` links `*s` to the data pointer `*data`, where `size` is the size of data element. The pointer `*data` is set to `NULL`, `s->n` and `s->alloc` are set to 0: the array is empty.

`void pari_stack_alloc(pari_stack *s, long nb)` makes room for `nb` more elements, i.e. makes sure that $s.alloc \geq s.n + nb$, possibly reallocating `data`.

`long pari_stack_new(pari_stack *s)` increases `s.n` by one unit, possibly reallocating `data`, and returns `s.n - 1`.

Caveat. The following construction is incorrect because `stack_new` can change the value of `t_foo`:

```
t_foo[ pari_stack_new(&s_foo) ] = x;
```

`void pari_stack_delete(pari_stack *s)` frees `data` and resets the stack to the state immediately following `stack_init` (`s->n` and `s->alloc` are set to 0).

`void * pari_stack_pushp(pari_stack *s, void *u)` This function assumes that `*data` is of pointer type. Pushes the element `u` on the stack `s`.

`void ** pari_stack_base(pari_stack *s)` returns the address of `data`, typecast to a `void **`.

11.7 Vectors and Matrices.

11.7.1 Access and extract. See Section 9.3.1 and Section 9.3.2 for various useful constructors. Coefficients are accessed and set using `gel`, `gcoeff`, see Section 5.2.7. There are many internal functions to extract or manipulate subvectors or submatrices but, like the accessors above, none of them are suitable for `gerepileupto`. Worse, there are no type verification, nor bound checking, so use at your own risk.

`GEN shallowcopy(GEN x)` returns a `GEN` whose components are the components of x (no copy is made). The result may now be used to compute in place without destroying x . This is essentially equivalent to

```
GEN y = cgetg(lg(x), typ(x));
for (i = 1; i < lg(x); i++) y[i] = x[i];
return y;
```

except that `t_MAT` is treated specially since shallow copies of all columns are made. The function also works for non-recursive types, but is useless in that case since it makes a deep copy. If x is known to be a `t_MAT`, you may call `RgM_shallowcopy` directly; if x is known not to be a `t_MAT`, you may call `leafcopy` directly.

`GEN RgM_shallowcopy(GEN x)` returns `shallowcopy(x)`, where x is a `t_MAT`.

`GEN shallowtrans(GEN x)` returns the transpose of x , *without* copying its components, i. e., it returns a `GEN` whose components are (physically) the components of x . This is the internal function underlying `gtrans`.

`GEN shallowconcat(GEN x, GEN y)` concatenate x and y , *without* copying components, i. e., it returns a `GEN` whose components are (physically) the components of x and y .

`GEN shallowconcat1(GEN x)` x must be `t_VEC` or `t_LIST`, concatenate its elements from left to right. Shallow version of `gconcat1`.

`GEN shallowmatconcat(GEN v)` shallow version of `matconcat`.

`GEN shallowextract(GEN x, GEN y)` extract components of the vector or matrix x according to the selection parameter y . This is the shallow analog of `extract0(x, y, NULL)`, see `vecextract`.

`GEN shallowmatextract(GEN M, GEN l1, GEN l2)` extract components of the matrix M according to the `t_VECSMALL` $l1$ (list of lines indices) and $l2$ (list of columns indices). This is the shallow analog of `extract0(x, l1, l2)`, see `vecextract`.

`GEN RgM_minor(GEN A, long i, long j)` given a square `t_MAT` A , return the matrix with i -th row and j -th column removed.

`GEN vconcat(GEN A, GEN B)` concatenate vertically the two `t_MAT` A and B of compatible dimensions. A `NULL` pointer is accepted for an empty matrix. See `shallowconcat`.

`GEN matslice(GEN A, long a, long b, long c, long d)` returns the submatrix $A[a..b, c..d]$. Assume $a \leq b$ and $c \leq d$.

`GEN row(GEN A, long i)` return $A[i,]$, the i -th row of the `t_MAT` A .

`GEN row_i(GEN A, long i, long j1, long j2)` return part of the i -th row of `t_MAT` A : $A[i, j_1]$, $A[i, j_1 + 1] \dots, A[i, j_2]$. Assume $j_1 \leq j_2$.

GEN rowcopy(GEN A, long i) return the row $A[i,]$ of the $\mathfrak{t_MAT}$ A. This function is memory clean and suitable for gerepileupto. See row for the shallow equivalent.

GEN rowslice(GEN A, long i1, long i2) return the $\mathfrak{t_MAT}$ formed by the i_1 -th through i_2 -th rows of $\mathfrak{t_MAT}$ A. Assume $i_1 \leq i_2$.

GEN rowsplice(GEN A, long i) return the $\mathfrak{t_MAT}$ formed from the coefficients of $\mathfrak{t_MAT}$ A with j -th row removed.

GEN rowpermute(GEN A, GEN p), p being a $\mathfrak{t_VECSMALL}$ representing a list $[p_1, \dots, p_n]$ of rows of $\mathfrak{t_MAT}$ A, returns the matrix whose rows are $A[p_1,], \dots, A[p_n,]$.

GEN rowslicepermute(GEN A, GEN p, long x1, long x2), short for

```
rowslice(rowpermute(A,p), x1, x2)
```

(more efficient).

GEN vecslice(GEN A, long j1, long j2), return $A[j_1], \dots, A[j_2]$. If A is a $\mathfrak{t_MAT}$, these correspond to *columns* of A. The object returned has the same type as A ($\mathfrak{t_VEC}$, $\mathfrak{t_COL}$ or $\mathfrak{t_MAT}$). Assume $j_1 \leq j_2$.

GEN vecsplice(GEN A, long j) return A with j -th entry removed ($\mathfrak{t_VEC}$, $\mathfrak{t_COL}$) or j -th column removed ($\mathfrak{t_MAT}$).

GEN vecreverse(GEN A). Returns a GEN which has the same type as A ($\mathfrak{t_VEC}$, $\mathfrak{t_COL}$ or $\mathfrak{t_MAT}$), and whose components are the $A[n], \dots, A[1]$. If A is a $\mathfrak{t_MAT}$, these are the *columns* of A.

void vecreverse_inplace(GEN A) as vecreverse, but reverse A in place.

GEN vecpermute(GEN A, GEN p) p is a $\mathfrak{t_VECSMALL}$ representing a list $[p_1, \dots, p_n]$ of indices. Returns a GEN which has the same type as A ($\mathfrak{t_VEC}$, $\mathfrak{t_COL}$ or $\mathfrak{t_MAT}$), and whose components are $A[p_1], \dots, A[p_n]$. If A is a $\mathfrak{t_MAT}$, these are the *columns* of A.

GEN vecsmallpermute(GEN A, GEN p) as vecpermute when A is a $\mathfrak{t_VECSMALL}$.

GEN vecslicepermute(GEN A, GEN p, long y1, long y2) short for

```
vecslice(vecpermute(A,p), y1, y2)
```

(more efficient).

11.7.2 Componentwise operations.

The following convenience routines automate trivial loops of the form

```
for (i = 1; i < lg(a); i++) gel(v,i) = f(gel(a,i), gel(b,i))
```

for suitable f :

GEN vecinv(GEN a). Given a vector a , returns the vector whose i -th component is $\text{ginv}(a[i])$.

GEN vecmul(GEN a, GEN b). Given a and b two vectors of the same length, returns the vector whose i -th component is $\text{gmul}(a[i], b[i])$.

GEN vecdiv(GEN a, GEN b). Given a and b two vectors of the same length, returns the vector whose i -th component is $\text{gdiv}(a[i], b[i])$.

GEN vecpow(GEN a, GEN n). Given n a $\mathfrak{t_INT}$, returns the vector whose i -th component is $a[i]^n$.

GEN `vecmodii`(GEN `a`, GEN `b`). Assuming `a` and `b` are two ZV of the same length, returns the vector whose i -th component is `modii(a[i], b[i])`.

GEN `vecmoduu`(GEN `a`, GEN `b`). Assuming `a` and `b` are two `t_VECSMALL` of the same length, returns the vector whose i -th component is `a[i] % b[i]`.

Note that `vecadd` or `vecsub` do not exist since `gadd` and `gsub` have the expected behavior. On the other hand, `ginv` does not accept vector types, hence `vecinv`.

11.7.3 Low-level vectors and columns functions.

These functions handle `t_VEC` as an abstract container type of GENs. No specific meaning is attached to the content. They accept both `t_VEC` and `t_COL` as input, but `col` functions always return `t_COL` and `vec` functions always return `t_VEC`.

Note. All the functions below are shallow.

GEN `const_col`(long `n`, GEN `x`) returns a `t_COL` of `n` components equal to `x`.

GEN `const_vec`(long `n`, GEN `x`) returns a `t_VEC` of `n` components equal to `x`.

int `vec_isconst`(GEN `v`) Returns 1 if all the components of `v` are equal, else returns 0.

void `vec_setconst`(GEN `v`, GEN `x`) `v` a pre-existing vector. Set all its components to `x`.

int `vec_is1to1`(GEN `v`) Returns 1 if the components of `v` are pair-wise distinct, i.e. if $i \mapsto v[i]$ is a 1-to-1 mapping, else returns 0.

GEN `vec_append`(GEN `V`, GEN `s`) append `s` to the vector `V`.

GEN `vec_prepend`(GEN `V`, GEN `s`) prepend `s` to the vector `V`.

GEN `vec_shorten`(GEN `v`, long `n`) shortens the vector `v` to `n` components.

GEN `vec_lengthen`(GEN `v`, long `n`) lengthens the vector `v` to `n` components. The extra components are not initialized.

GEN `vec_insert`(GEN `v`, long `n`, GEN `x`) inserts `x` at position `n` in the vector `v`.

11.8 Vectors of small integers.

11.8.1 t_VECSMALL.

These functions handle `t_VECSMALL` as an abstract container type of small signed integers. No specific meaning is attached to the content.

GEN `const_vecsmall`(long `n`, long `c`) returns a `t_VECSMALL` of `n` components equal to `c`.

GEN `vec_to_vecsmall`(GEN `z`) identical to `ZV_to_zv(z)`.

GEN `vecsmall_to_vec`(GEN `z`) identical to `zv_to_ZV(z)`.

GEN `vecsmall_to_col`(GEN `z`) identical to `zv_to_ZC(z)`.

GEN `vecsmall_to_vec_inplace`(GEN `z`) apply `stoi` to all entries of `z` and set its type to `t_VEC`.

GEN `vecsmall_copy`(GEN `x`) makes a copy of `x` on the stack.

GEN `vecsmall_shorten`(GEN `v`, long `n`) shortens the `t_VECSMALL` `v` to `n` components.

GEN `vecsmall_lengthen(GEN v, long n)` lengthens the `t_VECSMALL` `v` to `n` components. The extra components are not initialized.

GEN `vecsmall_indexsort(GEN x)` performs an indirect sort of the components of the `t_VECSMALL` `x` and return a permutation stored in a `t_VECSMALL`.

void `vecsmall_sort(GEN v)` sorts the `t_VECSMALL` `v` in place.

void `vecsmall_reverse(GEN v)` as `vecreverse` for a `t_VECSMALL` `v`.

long `vecsmall_max(GEN v)` returns the maximum of the elements of `t_VECSMALL` `v`, assumed non-empty.

long `vecsmall_indexmax(GEN v)` returns the index of the largest element of `t_VECSMALL` `v`, assumed non-empty.

long `vecsmall_min(GEN v)` returns the minimum of the elements of `t_VECSMALL` `v`, assumed non-empty.

long `vecsmall_indexmin(GEN v)` returns the index of the smallest element of `t_VECSMALL` `v`, assumed non-empty.

long `vecsmall_isin(GEN v, long x)` returns the first index `i` such that `v[i]` is equal to `x`. Naive search in linear time, does not assume that `v` is sorted.

GEN `vecsmall_uniq(GEN v)` given a `t_VECSMALL` `v`, return the vector of unique occurrences.

GEN `vecsmall_uniq_sorted(GEN v)` same as `vecsmall_uniq`, but assumes `v` sorted.

long `vecsmall_duplicate(GEN v)` given a `t_VECSMALL` `v`, return 0 if there is no duplicates, or the index of the first duplicate (`vecsmall_duplicate([1,1])` returns 2).

long `vecsmall_duplicate_sorted(GEN v)` same as `vecsmall_duplicate`, but assume `v` sorted.

int `vecsmall_lexcmp(GEN x, GEN y)` compares two `t_VECSMALL` lexically.

int `vecsmall_prefixcmp(GEN x, GEN y)` truncate the longest `t_VECSMALL` to the length of the shortest and compares them lexicographically.

GEN `vecsmall_prepend(GEN V, long s)` prepend `s` to the `t_VECSMALL` `V`.

GEN `vecsmall_append(GEN V, long s)` append `s` to the `t_VECSMALL` `V`.

GEN `vecsmall_concat(GEN u, GEN v)` concat the `t_VECSMALL` `u` and `v`.

long `vecsmall_coincidence(GEN u, GEN v)` returns the numbers of indices where `u` and `v` agree.

long `vecsmall_pack(GEN v, long base, long mod)` handles the `t_VECSMALL` `v` as the digit of a number in base `base` and return this number modulo `mod`. This can be used as an hash function.

GEN `vecsmall_prod(GEN v)` given a `t_VECSMALL` `v`, return the product of its entries.

11.8.2 Vectors of `t_VECSMALL`. These functions manipulate vectors of `t_VECSMALL` (`vecvecsmall`).

`GEN vecvecsmall_sort(GEN x)` sorts lexicographically the components of the vector `x`.

`GEN vecvecsmall_sort_uniq(GEN x)` sorts lexicographically the components of the vector `x`, removing duplicates entries.

`GEN vecvecsmall_indexsort(GEN x)` performs an indirect lexicographic sorting of the components of the vector `x` and return a permutation stored in a `t_VECSMALL`.

`long vecvecsmall_search(GEN x, GEN y, long flag)` `x` being a sorted `vecvecsmall` and `y` a `t_VECSMALL`, search `y` inside `x`. `flag` has the same meaning as for `setsearch`.

`GEN vecvecsmall_max(GEN x)` returns the largest entry in all $x[i]$, assumed non-empty.

Chapter 12: Functions related to the GP interpreter

12.1 Handling closures.

12.1.1 Functions to evaluate `t_CLOSURE`.

`void closure_disassemble(GEN C)` print the `t_CLOSURE C` in GP assembly format.

`GEN closure_callgenall(GEN C, long n, ...)` evaluate the `t_CLOSURE C` with the `n` arguments (of type `GEN`) following `n` in the function call. Assumes `C` has arity $\geq n$.

`GEN closure_callgenvec(GEN C, GEN args)` evaluate the `t_CLOSURE C` with the arguments supplied in the vector `args`. Assumes `C` has arity $\geq \lg(\text{args}) - 1$.

`GEN closure_callgenvecprec(GEN C, GEN args, long prec)` as `closure_callgenvec` but set the precision locally to `prec`.

`GEN closure_callgen1(GEN C, GEN x)` evaluate the `t_CLOSURE C` with argument `x`. Assumes `C` has arity ≥ 1 .

`GEN closure_callgen1prec(GEN C, GEN x, long prec)` as `closure_callgen1`, but set the precision locally to `prec`.

`GEN closure_callgen2(GEN C, GEN x, GEN y)` evaluate the `t_CLOSURE C` with argument `x, y`. Assumes `C` has arity ≥ 2 .

`void closure_callvoid1(GEN C, GEN x)` evaluate the `t_CLOSURE C` with argument `x` and discard the result. Assumes `C` has arity ≥ 1 .

The following technical functions are used to evaluate *inline* closures and closures of arity 0.

The control flow statements (`break`, `next` and `return`) will cause the evaluation of the closure to be interrupted; this is called below a *flow change*. When that occurs, the functions below generally return `NULL`. The caller can then adopt three positions:

- raises an exception (`closure_evalnobrk`).
- passes through (by returning `NULL` itself).
- handles the flow change.

`GEN closure_evalgen(GEN code)` evaluates a closure and returns the result, or `NULL` if a flow change occurred.

`GEN closure_evalnobrk(GEN code)` as `closure_evalgen` but raise an exception if a flow change occurs. Meant for iterators where interrupting the closure is meaningless, e.g. `intnum` or `sumnum`.

`void closure_evalvoid(GEN code)` evaluates a closure whose return value is ignored. The caller has to deal with eventual flow changes by calling `loop_break`.

The remaining functions below are for exceptional situations:

GEN `closure_evalres(GEN code)` evaluates a closure and returns the result. The difference with `closure_evalgen` being that, if the flow end by a `return` statement, the result will be the returned value instead of `NULL`. Used by the main GP loop.

GEN `closure_evalbrk(GEN code, long *status)` as `closure_evalres` but set `status` to a non-zero value if a flow change occurred. This variant is not stack clean. Used by the break loop.

GEN `closure_trapgen(long numerr, GEN code)` evaluates closure, while trapping error `numerr`. Return (GEN)1L if error trapped, and the result otherwise, or `NULL` if a flow change occurred. Used by trap.

12.1.2 Functions to handle control flow changes.

`long loop_break(void)` processes an eventual flow changes inside an iterator. If this function return 1, the iterator should stop.

12.1.3 Functions to deal with lexical local variables.

Function using the prototype code 'V' need to manually create and delete a lexical variable for each code 'V', which will be given a number $-1, -2, \dots$

`void push_lex(GEN a, GEN code)` creates a new lexical variable whose initial value is a on the top of the stack. This variable get the number -1 , and the number of the other variables is decreased by one unit. When the first variable of a closure is created, the argument `code` must be the closure that references this lexical variable. The argument `code` must be `NULL` for all subsequent variables (if any). (The closure contains the debugging data for the variable).

`void pop_lex(long n)` deletes the n topmost lexical variables, increasing the number of other variables by n . The argument n must match the number of variables allocated through `push_lex`.

GEN `get_lex(long vn)` get the value of the variable with number `vn`.

`void set_lex(long vn, GEN x)` set the value of the variable with number `vn`.

12.1.4 Functions returning new closures.

GEN `compile_str(const char *s)` returns the closure corresponding to the GP expression s .

GEN `closure_deriv(GEN code)` returns a closure corresponding to the numerical derivative of the closure code.

GEN `snm_closure(entree *ep, GEN data)` Let `data` be a vector of length m , `ep` be an entree pointing to a C function f of arity $n + m$, returns a `t_CLOSURE` object g of arity n such that $g(x_1, \dots, x_n) = f(x_1, \dots, x_n, gel(data, 1), \dots, gel(data, m))$. If `data` is `NULL`, then $m = 0$ is assumed. This function has a low overhead since it does not copy `data`.

GEN `strtofunction(char *str)` returns a closure corresponding to the built-in or install'ed function named `str`.

GEN `strtoclosure(char *str, long n, ...)` returns a closure corresponding to the built-in or install'ed function named `str` with the n last parameters set to the n GENs following n , see `snm_closure`. This function has an higher overhead since it copies the parameters and does more input validation.

In the example code below, `agm1` is set to the function `x->agm(x,1)` and `res` is set to `agm(2,1)`.

```
GEN agm1 = strtoclosure("agm",1, gen_1);
GEN res = closure_callgen1(agm1, gen_2);
```

12.1.5 Functions used by the gp debugger (break loop). `long closure_context(long s)` restores the compilation context starting at frame `s+1`, and returns the index of the topmost frame. This allow to compile expressions in the topmost lexical scope.

`void closure_err(long level)` prints a backtrace of the last 20 stack frames, starting at frame `level`, the numbering starting at 0.

12.1.6 Standard wrappers for iterators. Two families of standard wrappers are provided to interface iterators like `intnum` or `sumnum` with GP.

12.1.6.1 Standard wrappers for inline closures. These wrappers are used to implement GP functions taking inline closures as input. The object `(GEN)E` must be an inline closure which is evaluated with the lexical variable number `-1` set to `x`.

`GEN gp_eval(void *E, GEN x)` is used for the prototype code ‘E’.

`GEN gp_evalprec(void *E, GEN x, long prec)` as `gp_eval`, but set the precision locally to `prec`.

`long gp_evalvoid(void *E, GEN x)` is used for the prototype code ‘I’. The resulting value is discarded. Return a non-zero value if a control-flow instruction request the iterator to terminate immediately.

`long gp_evalbool(void *E, GEN x)` returns the boolean `gp_eval(E, x)` evaluates to (i.e. true iff the value is non-zero).

`GEN gp_evalupto(void *E, GEN x)` memory-safe version of `gp_eval`, `gcopy`-ing the result, when the evaluator returns components of previously allocated objects (e.g. member functions).

12.1.6.2 Standard wrappers for true closures. These wrappers are used to implement GP functions taking true closures as input.

`GEN gp_call(void *E, GEN x)` evaluates the closure `(GEN)E` on `x`.

`GEN gp_callprec(void *E, GEN x, long prec)` as `gp_call`, but set the precision locally to `prec`.

`GEN gp_call2(void *E, GEN x, GEN y)` evaluates the closure `(GEN)E` on `(x, y)`.

`long gp_callbool(void *E, GEN x)` evaluates the closure `(GEN)E` on `x`, returns 1 if its result is non-zero, and 0 otherwise.

`long gp_callvoid(void *E, GEN x)` evaluates the closure `(GEN)E` on `x`, discarding the result. Return a non-zero value if a control-flow instruction request the iterator to terminate immediately.

12.2 Defaults.

`entree* pari_is_default(const char *s)` return the `entree` structure attached to `s` if it is the name of a default, `NULL` otherwise.

GEN `setdefault(const char *s, const char *v, long flag)` is the low-level function underlying `default0`. If `s` is `NULL`, call all default setting functions with string argument `NULL` and flag `d_ACKNOWLEDGE`. Otherwise, check whether `s` corresponds to a default and call the corresponding default setting function with arguments `v` and `flag`.

We shall describe these functions below: if `v` is `NULL`, we only look at the default value (and possibly print or return it, depending on `flag`); otherwise the value of the default to `v`, possibly after some translation work. The flag is one of

- `d_INITRC` called while reading the `gprc`: print and return `gnil`, possibly defer until `gp` actually starts.
- `d_RETURN` return the current value, as a `t_INT` if possible, as a `t_STR` otherwise.
- `d_ACKNOWLEDGE` print the current value, return `gnil`.
- `d_SILENT` print nothing, return `gnil`.

Low-level functions called by `setdefault`:

GEN `sd_TeXstyle(const char *v, long flag)`

GEN `sd_breakloop(const char *v, long flag)`

GEN `sd_colors(const char *v, long flag)`

GEN `sd_compatible(const char *v, long flag)`

GEN `sd_datadir(const char *v, long flag)`

GEN `sd_debug(const char *v, long flag)`

GEN `sd_debugfiles(const char *v, long flag)`

GEN `sd_debugmem(const char *v, long flag)`

GEN `sd_echo(const char *v, long flag)`

GEN `sd_factor_add_primes(const char *v, long flag)`

GEN `sd_factor_proven(const char *v, long flag)`

GEN `sd_format(const char *v, long flag)`

GEN `sd_graphcolormap(const char *v, long flag)`

GEN `sd_graphcolors(const char *v, long flag)`

GEN `sd_help(const char *v, long flag)`

GEN `sd_histfile(const char *v, long flag)`

GEN `sd_histsize(const char *v, long flag)`

GEN `sd_lines(const char *v, long flag)`

GEN `sd_linewrap(const char *v, long flag)`

GEN sd_log(const char *v, long flag)
 GEN sd_logfile(const char *v, long flag)
 GEN sd_nbthreads(const char *v, long flag)
 GEN sd_new_galois_format(const char *v, long flag)
 GEN sd_output(const char *v, long flag)
 GEN sd_parisize(const char *v, long flag)
 GEN sd_parisizemax(const char *v, long flag)
 GEN sd_path(const char *v, long flag)
 GEN sd_plothsizes(const char *v, long flag)
 GEN sd_prettyprinter(const char *v, long flag)
 GEN sd_primelimit(const char *v, long flag)
 GEN sd_prompt(const char *v, long flag)
 GEN sd_prompt_cont(const char *v, long flag)
 GEN sd_psfile(const char *v, long flag) The psfile default is obsolete, don't use this function.
 GEN sd_readline(const char *v, long flag)
 GEN sd_realbitprecision(const char *v, long flag)
 GEN sd_realprecision(const char *v, long flag)
 GEN sd_recover(const char *v, long flag)
 GEN sd_secure(const char *v, long flag)
 GEN sd_seriesprecision(const char *v, long flag)
 GEN sd_simplify(const char *v, long flag)
 GEN sd_sopath(const char *v, int flag)
 GEN sd_strictargs(const char *v, long flag)
 GEN sd_strictmatch(const char *v, long flag)
 GEN sd_timer(const char *v, long flag)
 GEN sd_threadsize(const char *v, long flag)
 GEN sd_threadsizemax(const char *v, long flag)

Generic functions used to implement defaults: most of the above routines are implemented in terms of the following generic ones. In all routines below

- **v** and **flag** are the arguments passed to default: **v** is a new value (or the empty string: no change), and **flag** is one of `d_INITRC`, `d_RETURN`, etc.
- **s** is the name of the default being changed, used to display error messages or acknowledgements.

GEN sd_toggle(const char *v, long flag, const char *s, int *ptn)

- if v is neither "0" nor "1", an error is raised using pari_err.
- ptn points to the current numerical value of the toggle (1 or 0), and is set to the new value (when v is non-empty).

For instance, here is how the timer default is implemented internally:

```
GEN
sd_timer(const char *v, long flag)
{ return sd_toggle(v,flag,"timer", &(GP_DATA->chrono)); }
```

The exact behavior and return value depends on flag:

- d_RETURN: returns the new toggle value, as a GEN.
- d_ACKNOWLEDGE: prints a message indicating the new toggle value and return gnil.
- other cases: print nothing and return gnil.

GEN sd_ulong(const char *v, long flag, const char *s, ulong *ptn, ulong Min, ulong Max, const char **msg)

- ptn points to the current numerical value of the toggle, and is set to the new value (when v is non-empty).

- Min and Max point to the minimum and maximum values allowed for the default.

• v must translate to an integer in the allowed ranger, a suffix among k/K ($\times 10^3$), m/M ($\times 10^6$), or g/G ($\times 10^9$) is allowed, but no arithmetic expression.

• msg is a [NULL]-terminated array of messages or NULL (ignored). If msg is not NULL, msg[i] contains a message attached to the value i of the default. The last entry in the msg array is used as a message attached to all subsequent ones.

The exact behavior and return value depends on flag:

- d_RETURN: returns the new value, as a GEN.
- d_ACKNOWLEDGE: prints a message indicating the new value, possibly a message attached to it via the msg argument, and return gnil.
- other cases: print nothing and return gnil.

GEN sd_intarray(const char *v, long flag, const char *s, GEN *pz)

- records a t_VECSMALL array of non-negative integers.
- pz points to the current t_VECSMALL value, and is set to the new value (when v is non-empty).

The exact return value depends on flag:

- d_RETURN: returns the new value, as a t_VEC (converted via zv_to_ZV)
- d_ACKNOWLEDGE: prints a message indicating the new value, (as a t_VEC) and return gnil.
- other cases: print nothing and return gnil.

GEN sd_string(const char *v, long flag, const char *s, char **pstr) • v is subject to environment expansion, then time expansion.

- pstr points to the current string value, and is set to the new value (when v is non-empty).

12.3 Records and Lazy vectors.

The functions in this section are used to implement `ell` structures and analogous objects, which are vectors some of whose components are initialized to dummy values, later computed on demand. We start by initializing the structure:

`GEN obj_init(long d, long n)` returns an *obj* S , a `t_VEC` with d regular components, accessed as `gel(S,1), ..., gel(S,d)`; together with a record of n members, all initialized to 0. The arguments d and n must be non-negative.

After $S = \text{obj_init}(d, n)$, the prototype of our other functions are of the form

```
GEN obj_do(GEN S, long tag, ...)
```

The first argument S holds the structure to be managed. The second argument *tag* is the index of the struct member (from 1 to n) we operate on. We recommend to define an `enum` and use descriptive names instead of hardcoded numbers. For instance, if $n = 3$, after defining

```
enum { TAG_p = 1, TAG_list, TAG_data };
```

one may use `TAG_list` or 2 indifferently as a tag. The former being preferred, of course.

Technical note. In the current implementation, S is a `t_VEC` with $d + 1$ entries. The first d components are ordinary `t_GEN` entries, which you can read or assign to in the customary way. But the last component `gel(S,d + 1)`, a `t_VEC` of length n initialized to `zerovec(n)`, must be handled in a special way: you should never access or modify its components directly, only through the API we are about to describe. Indeed, its entries are meant to contain dynamic data, which will be stored, retrieved and replaced (for instance by a value computed to a higher accuracy), while interacting safely with intermediate `gerepile` calls. This mechanism allows to simulate C `structs`, in a simpler way than with general hashtables, while remaining compatible with the GP language, which knows neither `structs` nor hashtables. It also serialize the structure in an ordinary `GEN`, which facilitates copies and garbage collection (use `gcopy` or `gerepile`), rather than having to deal with individual components of actual C `structs`.

`GEN obj_reinit(GEN S)` make a shallow copy of S , re-initializing all dynamic components. This allows “forking” a lazy vector while avoiding both a memory leak, and storing pointers to the same data in different objects (with risks of a double free later).

`GEN obj_check(GEN S, long tag)` if the *tag*-component in S is non empty, return it. Otherwise return `NULL`. The `t_INT 0` (initial value) is used as a sentinel to indicated an empty component.

`GEN obj_insert(GEN S, long tag, GEN O)` insert (a clone of) O as *tag*-component of S . Any previous value is deleted, and data pointing to it become invalid.

`GEN obj_insert_shallow(GEN S, long K, GEN O)` as `obj_insert`, inserting O as-is, not via a clone.

`GEN obj_checkbuild(GEN S, long tag, GEN (*build)(GEN))` if the *tag*-component of S is non empty, return it. Otherwise insert (a clone of) `build(S)` as *tag*-component in S , and return it.

`GEN obj_checkbuild_padicprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of S is non empty *and* has relative p -adic precision $\geq \text{prec}$, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S , and return it.

`GEN obj_checkbuild_realprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of S is non empty *and* satisfies `gprecision` $\geq \text{prec}$, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in S , and return it.

GEN obj_checkbuild_prec(GEN S, long tag, GEN (*build)(GEN, long), GEN (*gpr)(GEN), long prec) if the *tag*-component of *S* is non empty *and* has precision $\text{gpr}(x) \geq \text{prec}$, return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in *S*, and return it.

void obj_free(GEN S) destroys all clones stored in the *n* tagged components, and replace them by the initial value 0. The regular entries of *S* are unaffected, and *S* remains a valid object. This is used to avoid memory leaks.

Chapter 13: Algebraic Number Theory

13.1 General Number Fields.

13.1.1 Number field types.

None of the following routines thoroughly check their input: they distinguish between *bona fide* structures as output by PARI routines, but designing perverse data will easily fool them. To give an example, a square matrix will be interpreted as an ideal even though the \mathbf{Z} -module generated by its columns may not be an \mathbf{Z}_K -module (i.e. the expensive `nfideal` routine will *not* be called).

`long nftyp(GEN x)`. Returns the type of number field structure stored in `x`, `typ_NF`, `typ_BNF`, or `typ_BNR`. Other answers are possible, meaning `x` is not a number field structure.

`GEN get_nf(GEN x, long *t)`. Extract an *nf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_bnf(GEN x, long *t)`. Extract a *bnf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_nfpol(GEN x, GEN *nf)` try to extract an *nf* structure from `x`, and sets `*nf` to `NULL` (failure) or to the *nf*. Returns the (monic, integral) polynomial defining the field.

`GEN get_bnfpol(GEN x, GEN *bnf, GEN *nf)` try to extract a *bnf* and an *nf* structure from `x`, and sets `*bnf` and `*nf` to `NULL` (failure) or to the corresponding structure. Returns the (monic, integral) polynomial defining the field.

`GEN checknf(GEN x)` if an *nf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_nf` is often more flexible.

`GEN checkbnf(GEN x)` if an *bnf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_bnf` is often more flexible.

`GEN checkbnf_i(GEN bnf)` same as `checkbnf` but return `NULL` instead of raising an exception.

`void checkbnr(GEN bnr)` Raise an exception if the argument is not a *bnr* structure.

`GEN checknf_i(GEN nf)` same as `checknf` but return `NULL` instead of raising an exception.

`void checkbnrgen(GEN bnr)` Raise an exception if the argument is not a *bnr* structure, complete with explicit generators for the ray class group. This is normally useless and `checkbnr` should be instead, unless you are absolutely certain that the generators will be needed at a later point, and you are about to embark in a costly intermediate computation. PARI functions do check that generators are present in *bnr* before accessing them: they will raise an error themselves; many functions that may require them, e.g. `bnrconductor`, often do not actually need them.

`void checkrnf(GEN rnf)` Raise an exception if the argument is not an *rnf* structure.

`int checkrnf_i(GEN rnf)` same as `checkrnf` but return 0 on failure and 1 on success.

`void checkbid(GEN bid)` Raise an exception if the argument is not a *bid* structure.

`GEN checkbid_i(GEN bid)` same as `checkbid` but return NULL instead of raising an exception and return *bid* on success.

`GEN checkznstar_i(GEN G)` return *G* if it is a *znstar*; else return NULL on failure.

`GEN checkgal(GEN x)` if a *galoisinit* structure can be extracted from *x*, return it; otherwise raise an exception.

`void checksqmat(GEN x, long N)` check whether *x* is a square matrix of dimension *N*. May be used to check for ideals if *N* is the field degree.

`void checkprid(GEN pr)` Raise an exception if the argument is not a prime ideal structure.

`int checkprid_i(GEN pr)` same as `checkprid` but return 0 instead of raising an exception and return 1 on success.

`int is_nf_factor(GEN F)` return 1 if *F* is an ideal factorization and 0 otherwise.

`int is_nf_extfactor(GEN F)` return 1 if *F* is an extended ideal factorization (allowing 0 or negative exponents) and 0 otherwise.

`GEN get_prid(GEN ideal)` return the underlying prime ideal structure if one can be extracted from *ideal* (ideal or extended ideal), and return NULL otherwise.

`void checkabgrp(GEN v)` Raise an exception if the argument is not an abelian group structure, i.e. a `t_VEC` with either 2 or 3 entries: $[N, cyc]$ or $[N, cyc, gen]$.

`GEN abgrp_get_no(GEN x)` extract the cardinality *N* from an abelian group structure.

`GEN abgrp_get_cyc(GEN x)` extract the elementary divisors *cyc* from an abelian group structure.

`GEN abgrp_get_gen(GEN x)` extract the generators *gen* from an abelian group structure.

`void checkmodpr(GEN modpr)` Raise an exception if the argument is not a `modpr` structure (from `nfmodprinit`).

`GEN get_modpr(GEN x)` return *x* if it is a `modpr` structure and NULL otherwise.

`GEN checknfelt_mod(GEN nf, GEN x, const char *s)` given an *nf* structure *nf* and a `t_POLMOD` *x*, return the attached polynomial representative (shallow) if *x* and *nf* are compatible. Raise an exception otherwise. Set *s* to the name of the caller for a meaningful error message.

`void check_ZKmodule(GEN x, const char *s)` check whether *x* looks like \mathbf{Z}_K -module (a pair $[A, I]$, where *A* is a matrix and *I* is a list of ideals; *A* has as many columns as *I* has elements. Otherwise raises an exception. Set *s* to the name of the caller for a meaningful error message.

`long idealtyp(GEN *ideal, GEN *fa)` The input is *ideal*, a pointer to an ideal (or extended ideal), which is usually modified, *fa* being set as a side-effect. Returns the type of the underlying ideal among `id_PRINCIPAL` (a number field element), `id_PRIME` (a prime ideal) `id_MAT` (an ideal in matrix form).

If *ideal* pointed to an ideal, set *fa* to NULL, and possibly simplify *ideal* (for instance the zero ideal is replaced by `gen_0`). If it pointed to an extended ideal, replace *ideal* by the underlying ideal and set *fa* to the factorization matrix component.

13.1.2 Extracting info from a nf structure.

These functions expect a true *nf* argument attached to a number field $K = \mathbf{Q}[x]/(T)$, e.g. a *bnf* will not work. Let $n = [K : \mathbf{Q}]$ be the field degree.

`GEN nf_get_pol(GEN nf)` returns the polynomial T (monic, in $\mathbf{Z}[x]$).

`long nf_get_varn(GEN nf)` returns the variable number of the number field defining polynomial.

`long nf_get_r1(GEN nf)` returns the number of real places r_1 .

`long nf_get_r2(GEN nf)` returns the number of complex places r_2 .

`void nf_get_sign(GEN nf, long *r1, long *r2)` sets r_1 and r_2 to the number of real and complex places respectively. Note that $r_1 + 2r_2$ is the field degree.

`long nf_get_degree(GEN nf)` returns the number field degree, $n = r_1 + 2r_2$.

`GEN nf_get_disc(GEN nf)` returns the field discriminant.

`GEN nf_get_index(GEN nf)` returns the index of T , i.e. the index of the order generated by the power basis $(1, x, \dots, x^{n-1})$ in the maximal order of K .

`GEN nf_get_zk(GEN nf)` returns a basis (w_1, w_2, \dots, w_n) for the maximal order of K . Those are polynomials in $\mathbf{Q}[x]$ of degree $< n$; it is guaranteed that $w_1 = 1$.

`GEN nf_get_zkden(GEN nf)` returns the denominator of `nf_get_zk`, as a positive `t_INT`.

`GEN nf_get_zkprimpart(GEN nf)` returns `nf_get_zk` times its denominator.

`GEN nf_get_invzk(GEN nf)` returns the matrix $(m_{i,j}) \in M_n(\mathbf{Z})$ giving the power basis (x^i) in terms of the (w_j) , i.e. such that $x^{j-1} = \sum_{i=1}^n m_{i,j} w_i$ for all $1 \leq j \leq n$; since $w_1 = 1 = x^0$, we have $m_{i,1} = \delta_{i,1}$ for all i . The conversion functions in the `algtobasis` family essentially amount to a left multiplication by this matrix.

`GEN nf_get_roots(GEN nf)` returns the r_1 real roots of the polynomial defining the number fields: first the r_1 real roots (as `t_REALS`), then the r_2 representatives of the pairs of complex conjugates.

`GEN nf_get_allroots(GEN nf)` returns all the complex roots of T : first the r_1 real roots (as `t_REALS`), then the r_2 pairs of complex conjugates.

`GEN nf_get_M(GEN nf)` returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K : $M[i, j]$ contains $w_j(\alpha_i)$, where α_i is the i -th element of `nf_get_roots(nf)`. In particular, if v is an n -th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i] w_i$ of K , then `RgM_RgC_mul(M, v)` represents the embeddings of v .

`GEN nf_get_G(GEN nf)` returns a $n \times n$ real matrix G such that $Gv \cdot Gv = T_2(v)$, where v is an n -th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i] w_i$ of K and T_2 is the standard Euclidean form on $K \otimes \mathbf{R}$, i.e. $T_2(v) = \sum_{\sigma} |\sigma(v)|^2$, where σ runs through all n complex embeddings of K .

`GEN nf_get_roundG(GEN nf)` returns a rescaled version of G , rounded to nearest integers, specifically `RM_round_maxrank(G)`.

`GEN nf_get_ramified_primes(GEN nf)` returns the vector of ramified primes.

`GEN nf_get_Tr(GEN nf)` returns the matrix of the Trace quadratic form on the basis (w_1, \dots, w_n) : its (i, j) entry is $\text{Tr} w_i w_j$.

`GEN nf_get_diff(GEN nf)` returns the primitive part of the inverse of the above Trace matrix.

`long nf_get_prec(GEN nf)` returns the precision (in words) to which the *nf* was computed.

13.1.3 Extracting info from a bnf structure.

These functions expect a true *bnf* argument, e.g. a *bnr* will not work.

GEN `bnf_get_nf`(GEN `bnf`) returns the underlying *nf*.

GEN `bnf_get_clgp`(GEN `bnf`) returns the class group in *bnf*, which is a 3-component vector $[h, cyc, gen]$.

GEN `bnf_get_cyc`(GEN `bnf`) returns the elementary divisors of the class group (cyclic components) $[d_1, \dots, d_k]$, where $d_k \mid \dots \mid d_1$.

GEN `bnf_get_gen`(GEN `bnf`) returns the generators $[g_1, \dots, g_k]$ of the class group. Each g_i has order d_i , and the full module of relations between the g_i is generated by the $d_i g_i = 0$.

GEN `bnf_get_no`(GEN `bnf`) returns the class number.

GEN `bnf_get_reg`(GEN `bnf`) returns the regulator.

GEN `bnf_get_logfu`(GEN `bnf`) returns (complex floating point approximations to) the logarithms of the complex embeddings of our system of fundamental units.

GEN `bnf_get_fu`(GEN `bnf`) returns the fundamental units. Raise an error if the *bnf* does not contain units in algebraic form.

GEN `bnf_get_fu_nocheck`(GEN `bnf`) as `bnf_get_fu` without checking whether units are present. Do not use this unless you initialize the *bnf* yourself!

GEN `bnf_get_tuU`(GEN `bnf`) returns a generator of the torsion part of \mathbf{Z}_K^* .

long `bnf_get_tuN`(GEN `bnf`) returns the order of the torsion part of \mathbf{Z}_K^* , i.e. the number of roots of unity in K .

13.1.4 Extracting info from a bnr structure.

These functions expect a true *bnr* argument.

GEN `bnr_get_bnf`(GEN `bnr`) returns the underlying *bnf*.

GEN `bnr_get_nf`(GEN `bnr`) returns the underlying *nf*.

GEN `bnr_get_clgp`(GEN `bnr`) returns the ray class group.

GEN `bnr_get_no`(GEN `bnr`) returns the ray class number.

GEN `bnr_get_cyc`(GEN `bnr`) returns the elementary divisors of the ray class group (cyclic components) $[d_1, \dots, d_k]$, where $d_k \mid \dots \mid d_1$.

GEN `bnr_get_gen`(GEN `bnr`) returns the generators $[g_1, \dots, g_k]$ of the ray class group. Each g_i has order d_i , and the full module of relations between the g_i is generated by the $d_i g_i = 0$. Raise a generic error if the *bnr* does not contain the ray class group generators.

GEN `bnr_get_gen_nocheck`(GEN `bnr`) as `bnr_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bnr* yourself!

GEN `bnr_get_bid`(GEN `bnr`) returns the *bid* attached to the *bnr* modulus.

GEN `bnr_get_mod`(GEN `bnr`) returns the modulus attached to the *bnr*.

13.1.5 Extracting info from an rnf structure.

These functions expect a true *rnf* argument, attached to an extension L/K , $K = \mathbf{Q}[y]/(T)$, $L = K[x]/(P)$.

`long rnf_get_degree(GEN rnf)` returns the *relative* degree $[L : K]$.

`long rnf_get_absdegree(GEN rnf)` returns the absolute degree $[L : \mathbf{Q}]$.

`long rnf_get_nfdegree(GEN rnf)` returns the degree of the base field $[K : \mathbf{Q}]$.

`GEN rnf_get_nf(GEN rnf)` returns the base field K , an *nf* structure.

`GEN rnf_get_nfpol(GEN rnf)` returns the polynomial T defining the base field K .

`long rnf_get_nfvarn(GEN rnf)` returns the variable y attached to the base field K .

`GEN rnf_get_nfzk(GEN rnf)` returns the integer basis of the base field K .

`GEN rnf_get_pol(GEN rnf)` returns the relative polynomial defining L/K .

`long rnf_get_varn(GEN rnf)` returns the variable x attached to L .

`GEN rnf_get_zk(GEN nf)` returns the relative integer basis generating \mathbf{Z}_L as a \mathbf{Z}_K -module, as a pseudo-matrix (A, I) in HNF.

`GEN rnf_get_disc(GEN rnf)` is the output $[\mathfrak{d}, s]$ of `rnfdisc`.

`GEN rnf_get_idealdisc(GEN rnf)` is the ideal discriminant \mathfrak{d} from `rnfdisc`.

`GEN rnf_get_index(GEN rnf)` is the index ideal \mathfrak{f}

`GEN rnf_get_polabs(GEN rnf)` returns an absolute polynomial defining L/\mathbf{Q} .

`GEN rnf_get_alpha(GEN rnf)` a root α of the polynomial defining the base field, modulo `polabs` (cf. `rnfequation`)

`GEN rnf_get_k(GEN rnf)` a small integer k such that $\theta = \beta + k\alpha$ is a root of `polabs`, where β is a root of `pol` and α a root of the polynomial defining the base field, as in `rnf_get_alpha` (cf. also `rnfequation`).

`GEN rnf_get_invzk(GEN rnf)` contains A^{-1} , where (A, I) is the chosen pseudo-basis for \mathbf{Z}_L over \mathbf{Z}_K .

`GEN rnf_get_map(GEN rnf)` returns technical data attached to the map $K \rightarrow L$. Currently, this contains data from `rnfequation`, as well as the polynomials T and P .

13.1.6 Extracting info from a bid structure.

These functions expect a true *bid* argument, attached to a modulus $I = I_0 I_\infty$ in a number field K .

`GEN bid_get_mod(GEN bid)` returns the modulus attached to the *bid*.

`GEN bid_get_grp(GEN bid)` returns the Abelian group attached to $(\mathbf{Z}_K/I)^*$.

`GEN bid_get_ideal(GEN bid)` return the finite part I_0 of the *bid* modulus (an integer ideal).

`GEN bid_get_arch(GEN bid)` return the Archimedean part I_∞ of the *bid* modulus as a vector of real places in `vec01` format, see Section 13.1.17.

GEN `bid_get_archp`(GEN `bid`) return the Archimedean part I_∞ of the `bid` modulus, as a vector of real places in indices format see Section 13.1.17.

GEN `bid_get_fact`(GEN `bid`) returns the ideal factorization $I_0 = \prod_i \mathfrak{p}_i^{e_i}$.

`bid_get_ideal`(`bid`), via `idealfactor`.

GEN `bid_get_no`(GEN `bid`) returns the cardinality of the group $(\mathbf{Z}_K/I)^*$.

GEN `bid_get_cyc`(GEN `bid`) returns the elementary divisors of the group $(\mathbf{Z}_K/I)^*$ (cyclic components) $[d_1, \dots, d_k]$, where $d_k \mid \dots \mid d_1$.

GEN `bid_get_gen`(GEN `bid`) returns the generators of $(\mathbf{Z}_K/I)^*$ contained in `bid`. Raise a generic error if `bid` does not contain generators.

GEN `bid_get_gen_nocheck`(GEN `bid`) as `bid_get_gen` without checking whether generators are present. Do not use this unless you initialize the `bid` yourself!

GEN `bid_get_sprk`(GEN `bid`) return a list of structures attached to the $(\mathbf{Z}_K/\mathfrak{p}^e)^*$ where \mathfrak{p}^e divides I_0 exactly.

GEN `bid_get_sarch`(GEN `bid`) return the structure attached to $(\mathbf{Z}_K/I_\infty)^*$, by `nfarchstar`.

GEN `bid_get_U`(GEN `bid`) return the matrix with integral coefficients relating the local generators (from chinese remainders) to the global SNF generators (`bid.gen`).

13.1.7 Extracting info from a znstar structure.

These functions expect an argument G as returned by `znstar0`(N , 1), attached to a positive N and the abelian group $(\mathbf{Z}/N\mathbf{Z})^*$. Let (g_i) be the SNF generators, where g_i has order d_i ; we call (g'_i) the (canonical) Conrey generators, where g'_i has order d'_i . Both sets of generators have the same cardinality.

GEN `znstar_get_N`(GEN `bid`) return N .

GEN `znstar_get_faN`(GEN G) return the factorization `factor`(N), $N = \prod_j p_j^{e_j}$.

GEN `znstar_get_pe`(GEN G) return the vector of primary factors $(p_j^{e_j})$.

GEN `znstar_get_no`(GEN G) the cardinality $\phi(N)$ of G .

GEN `znstar_get_cyc`(GEN G) elementary divisors (d_i) of $(\mathbf{Z}/N\mathbf{Z})^*$.

GEN `znstar_get_gen`(GEN G) SNF generators divisors (g_i) of $(\mathbf{Z}/N\mathbf{Z})^*$.

GEN `znstar_get_conreycyc`(GEN G) orders (d'_i) of Conrey generators.

GEN `znstar_get_conreygen`(GEN G) Conrey generators (g'_i) .

GEN `znstar_get_U`(GEN G) a square matrix U such that $(g_i) = U(g'_i)$.

GEN `znstar_get_Ui`(GEN G) a square matrix U' such that $U'(g_i) = (g'_i)$. In general, UU' will not be the identity.

13.1.8 Inserting info in a number field structure.

If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

These functions expect a `bnf` argument:

`GEN bnf_build_cycgen(GEN bnf)` the *bnf* contains generators $[g_1, \dots, g_k]$ of the class group, each with order d_i . Then $g_i^{d_i} = (x_i)$ is a principal ideal. This function returns the x_i as a factorization matrix (`famat`) giving the element in factored form as a product of S -units.

`GEN bnf_build_matalpha(GEN bnf)` the class group was computed using a factorbase S of prime ideals $\mathfrak{p}_i, i \leq r$. They satisfy relations of the form $\prod_j \mathfrak{p}_i^{e_{i,j}} = (\alpha_j)$, where the $e_{i,j}$ are given by the matrices `bnf[1]` (W , singling out a minimal set of generators in S) and `bnf[2]` (B , expressing the rest of S in terms of the singled out generators). This function returns the α_j in factored form as a product of S -units.

`GEN bnf_build_units(GEN bnf)` returns a minimal set of generators for the unit group. The first element is a torsion unit, the others have infinite order.

These functions expect a `rnf` argument:

`GEN rnf_build_nfabs(GEN rnf, long prec)` given a *rnf* structure attached to L/K , (compute and) return an *nf* structure attached to L at precision `prec`.

`void rnfcomplete(GEN rnf)` as `rnf_build_nfabs` using the precision of K for `prec`.

`GEN rnf_zkabs(GEN rnf)` returns a \mathbf{Z} -basis in HNF for \mathbf{Z}_L as a pair $[T, v]$, where T is `rnf_get_polabs(rnf)` and v a vector of elements lifted from $\mathbf{Q}[X]/(T)$. Note that the function `rnf_build_nfabs` essentially applies `nfinit` to the output of this function.

13.1.9 Increasing accuracy.

`GEN nfnewprec(GEN x, long prec)`. Raise an exception if x is not a number field structure (*nf*, *bnf* or *bnr*). Otherwise, sets its accuracy to `prec` and return the new structure. This is mostly useful with `prec` larger than the accuracy to which x was computed, but it is also possible to decrease the accuracy of x (truncating relevant components, which may speed up later computations). This routine may modify the original x (see below).

This routine is straightforward for *nf* structures, but for the other ones, it requires all principal ideals corresponding to the *bnf* relations in algebraic form (they are originally only available via floating point approximations). This in turn requires many calls to `bnfisprincipal0`, which is often slow, and may fail if the initial accuracy was too low. In this case, the routine will not actually fail but recomputes a *bnf* from scratch!

Since this process may be very expensive, the corresponding data is cached (as a *clone*) in the *original* x so that later precision increases become very fast. In particular, the copy returned by `nfnewprec` also contains this additional data.

`GEN bnfnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnf* structure from x before increasing its accuracy, and returns only the latter.

`GEN bnrnewprec(GEN x, long prec)`. As `nfnewprec`, but extracts a *bnr* structure from x before increasing its accuracy, and returns only the latter.

`GEN nfnewprec_shallow(GEN nf, long prec)`

GEN bnfnewprec_shallow(GEN bnf, long prec)

GEN bnrnewprec_shallow(GEN bnr, long prec) Shallow functions underlying the above, except that the first argument must now have the corresponding number field type. I.e. one cannot call `nfnewprec_shallow(nf, prec)` if `nf` is actually a *bnf*.

13.1.10 Number field arithmetic. The number field $K = \mathbf{Q}[X]/(T)$ is represented by an `nf` (or `bnf` or `bnr` structure). An algebraic number belonging to K is given as

- a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo T), or
- a `t_POLMOD` (modulo T), or
- a `t_COL` v of dimension $N = [K : \mathbf{Q}]$, representing the element in terms of the computed integral basis (e_i) , as

$$\text{sum}(i = 1, N, v[i] * nf.zk[i])$$

The preferred forms are `t_INT` and `t_COL` of `t_INT`. Routines can handle denominators but it is much more efficient to remove denominators first (`Q_remove_denom`) and take them into account at the end.

Safe routines. The following routines do not assume that their `nf` argument is a true *nf* (it can be any number field type, e.g. a *bnf*), and accept number field elements in all the above forms. They return their result in `t_COL` form.

GEN nfadd(GEN nf, GEN x, GEN y) returns $x + y$.

GEN nfsub(GEN nf, GEN x, GEN y) returns $x - y$.

GEN nfdiv(GEN nf, GEN x, GEN y) returns x/y .

GEN nfinv(GEN nf, GEN x) returns x^{-1} .

GEN nfmul(GEN nf, GEN x, GEN y) returns xy .

GEN nfpow(GEN nf, GEN x, GEN k) returns x^k , k is in \mathbf{Z} .

GEN nfpow_u(GEN nf, GEN x, ulong k) returns x^k , $k \geq 0$.

GEN nfsqr(GEN nf, GEN x) returns x^2 .

long nfval(GEN nf, GEN x, GEN pr) returns the valuation of x at the maximal ideal \mathfrak{p} attached to the *prid* `pr`. Returns `LONG_MAX` if x is 0.

GEN nfnorm(GEN nf, GEN x) absolute norm of x .

GEN nftrace(GEN nf, GEN x) absolute trace of x .

GEN nfpoleval(GEN nf, GEN pol, GEN a) evaluate the `t_POL` `pol` (with coefficients in `nf`) on the algebraic number a (also in *nf*).

GEN FpX_FpC_nfpoleval(GEN nf, GEN pol, GEN a, GEN p) evaluate the `FpX` `pol` on the algebraic number a (also in *nf*).

The following three functions implement trivial functionality akin to Euclidean division for which we currently have no real use. Of course, even if the number field is actually Euclidean, these do not in general implement a true Euclidean division.

GEN `nfdiveuc`(GEN `nf`, GEN `a`, GEN `b`) returns the algebraic integer closest to x/y . Functionally identical to `ground(nfdiv(nf,x,y))`.

GEN `nfdivrem`(GEN `nf`, GEN `a`, GEN `b`) returns the vector $[q, r]$, where

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));    \\ or r = nfmod(nf,a,b);
```

GEN `nfmod`(GEN `nf`, GEN `a`, GEN `b`) returns r such that

```
q = nfdiveuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));
```

GEN `nf_to_scalar_or_basis`(GEN `nf`, GEN `x`) let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its basis representation (`nfalgtobasis`). Shallow function.

GEN `nf_to_scalar_or_alg`(GEN `nf`, GEN `x`) let x be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its lifted `t_POLMOD` representation (`lifted_nfbasistoalg`). Shallow function.

GEN `RgX_to_nfX`(GEN `nf`, GEN `x`) let x be a `t_POL` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new polynomial. Shallow function.

GEN `RgM_to_nfM`(GEN `nf`, GEN `x`) let x be a `t_MAT` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new matrix. Shallow function.

GEN `RgC_to_nfC`(GEN `nf`, GEN `x`) let x be a `t_COL` or `t_VEC` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new `t_COL`. Shallow function.

Unsafe routines. The following routines assume that their `nf` argument is a true `nf` (e.g. a `bnf` is not allowed) and their argument are restricted in various ways, see the precise description below.

GEN `nfinvmodideal`(GEN `nf`, GEN `x`, GEN `A`) given an algebraic integer x and a non-zero integral ideal A in HNF, returns a y such that $xy \equiv 1$ modulo A .

GEN `nfpowmodideal`(GEN `nf`, GEN `x`, GEN `n`, GEN `ideal`) given an algebraic integer x , an integer n , and a non-zero integral ideal A in HNF, returns an algebraic integer congruent to x^n modulo A .

GEN `nfmuli`(GEN `nf`, GEN `x`, GEN `y`) returns $x \times y$ assuming that both x and y are either `t_INTs` or `ZVs` of the correct dimension.

GEN `nfsqri`(GEN `nf`, GEN `x`) returns x^2 assuming that x is a `t_INT` or a `ZV` of the correct dimension.

GEN `nfC_nf_mul`(GEN `nf`, GEN `v`, GEN `x`) given a `t_VEC` or `t_COL` v of elements of K in `t_INT`, `t_FRAC` or `t_COL` form, multiply it by the element x (arbitrary form). This is faster than multiplying coordinatewise since pre-computations related to x (computing the multiplication table) are done only once. The components of the result are in most cases `t_COLs` but are allowed to be `t_INTs` or `t_FRACs`. Shallow function.

GEN `nfC_multable_mul`(GEN `v`, GEN `mx`) same as `nfC_nf_mul`, where the argument x is replaced by its multiplication table `mx`.

GEN `zkC_multable_mul`(GEN `v`, GEN `x`) same as `nfC_nf_mul`, where v is a vector of algebraic integers, x is an algebraic integer, and x is replaced by `zk_multable(x)`.

GEN `zk_multable(GEN nf, GEN x)` given a ZC x (implicitly representing an algebraic integer), returns the ZM giving the multiplication table by x . Shallow function (the first column of the result points to the same data as x).

GEN `zk_inv(GEN nf, GEN x)` given a ZC x (implicitly representing an algebraic integer), returns the QC giving the inverse x^{-1} . Return NULL if x is 0. Not memory clean but safe for `gerepileupto`.

GEN `zkmultable_inv(GEN mx)` as `zk_inv`, where the argument given is `zk_multable(x)`.

GEN `zkmultable_capZ(GEN mx)` given a non-zero *zkmultable* mx attached to $x \in \mathbf{Z}_K$, return the positive generator of $(x) \cap \mathbf{Z}$.

GEN `zk_scalar_or_multable(GEN nf, GEN x)` given a `t_INT` or ZC x , returns a `t_INT` equal to x if the latter is a scalar (`t_INT` or `ZV_isscalar(x)` is 1) and `zk_multable(nf, x)` otherwise. Shallow function.

The following routines implement multiplication in a commutative R -algebra, generated by $(e_1 = 1, \dots, e_n)$, and given by a multiplication table M : elements in the algebra are n -dimensional `t_COLS`, and the matrix M is such that for all $1 \leq i, j \leq n$, its column with index $(i-1)n + j$, say (c_k) , gives $e_i \cdot e_j = \sum c_k e_k$. It is assumed that e_1 is the neutral element for the multiplication (a convenient optimization, true in practice for all multiplications we needed to implement). If x has any other type than `t_COL` where an algebra element is expected, it is understood as $x e_1$.

GEN `multable(GEN M, GEN x)` given a column vector x , representing the quantity $\sum_{i=1}^N x_i e_i$, returns the multiplication table by x . Shallow function.

GEN `ei_multable(GEN M, long i)` returns the multiplication table by the i -th basis element e_i . Shallow function.

GEN `tablemul(GEN M, GEN x, GEN y)` returns $x \cdot y$.

GEN `tablesqr(GEN M, GEN x)` returns x^2 .

GEN `tablemul_ei(GEN M, GEN x, long i)` returns $x \cdot e_i$.

GEN `tablemul_ei_ej(GEN M, long i, long j)` returns $e_i \cdot e_j$.

GEN `tablemulvec(GEN M, GEN x, GEN v)` given a vector v of elements in the algebra, returns the $x \cdot v[i]$.

The following routines implement naive linear algebra using the *black box field* mechanism:

GEN `nfM_det(GEN nf, GEN M)`

GEN `nfM_inv(GEN nf, GEN M)`

GEN `nfM_mul(GEN nf, GEN A, GEN B)`

GEN `nfM_nfc_mul(GEN nf, GEN A, GEN B)`

The following routines implement modular algorithms in cyclotomic fields. In the prototypes, P is the n -th cyclotomic polynomial Φ_n and M is a `t_MAT` with `t_INT` or `ZX` coefficients, understood modulo P .

GEN `ZabM_ker(GEN M, GEN P, long n)` returns an integral (primitive) basis of the kernel of M .

GEN `ZabM_indexrank(GEN M, GEN P, long n)` return a vector with two `t_VECSMALL` components givin the rank profile of M . Inefficient (but correct) when M does not have almost full column rank.

GEN `ZabM_inv`(GEN `M`, GEN `P`, long `n`, GEN `*pden`) assume that M is invertible; return N and sets the algebraic integer `*pden` (an integer or a ZX, implicitly modulo P) such that $MN = \text{den} \cdot \text{Id}$.

GEN `ZabM_pseudoinv`(GEN `M`, GEN `P`, long `n`, GEN `*pv`, GEN `*pden`) analog of `ZM_pseudoinv`. Not gerepile-safe.

GEN `ZabM_inv_ratlift`(GEN `M`, GEN `P`, long `n`, GEN `*pden`) return a primitive matrix H such that MH is d times the identity and set `*pden` to d . Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of M^{-1} is much smaller than $\det M$ else use `ZabM_inv`.

13.1.11 Elements in factored form.

Computational algebraic theory performs extensively linear algebra on \mathbf{Z} -modules with a natural multiplicative structure (K^* , fractional ideals in K , \mathbf{Z}_K^* , ideal class group), thereby raising elements to horrendously large powers. A seemingly innocuous elementary linear algebra operation like $C_i \leftarrow C_i - 10000C_1$ involves raising entries in C_1 to the 10000-th power. Understandably, it is often more efficient to keep elements in factored form rather than expand every such expression. A *factorization matrix* (or *famat*) is a two column matrix, the first column containing *elements* (arbitrary objects which may be repeated in the column), and the second one contains *exponents* (`t_INTs`, allowed to be 0). By abuse of notation, the empty matrix `cgetg(1, t_MAT)` is recognized as the trivial factorization (no element, no exponent).

Even though we think of a *famat* with columns g and e as one meaningful object when fully expanded as $\prod g^{e[i]}$, *famats* are basically about concatenating information to keep track of linear algebra: the objects stored in a *famat* need not be operation-compatible, they will not even be compared to each other (with one exception: `famat_reduce`). Multiplying two *famats* just concatenates their elements and exponents columns. In a context where a *famat* is expected, an object x which is not of type `t_MAT` will be treated as the factorization x^1 . The following functions all return *famats*:

GEN `famat_mul`(GEN `f`, GEN `g`) f, g are *famat*, or objects whose type is *not* `t_MAT` (understood as f^1 or g^1). Returns fg . The empty factorization is the neutral element for *famat* multiplication.

GEN `famat_mul_shallow`(GEN `f`, GEN `g`) shallow version of `famat_mul`.

GEN `famat_pow`(GEN `f`, GEN `n`) n is a `t_INT`. If f is a `t_MAT`, assume it is a *famat* and return f^n (multiplies the exponent column by n). Otherwise, understand it as an element and returns the 1-line *famat* f^n .

GEN `famat_pow_shallow`(GEN `f`, GEN `n`) shallow version of `famat_pow`.

GEN `famat_pows_shallow`(GEN `f`, long `n`) shallow version of `famat_pow` where n is a small integer.

GEN `famat_mulpow_shallow`(GEN `f`, GEN `g`, GEN `e`) *famat* corresponding to $f \cdot g^e$. Shallow function.

GEN `famat_mulpows_shallow`(GEN `f`, GEN `g`, long `e`) *famat* shallow version of `famat_mulpow` where e is a small integer.

GEN `famat_sqr`(GEN `f`) returns f^2 .

GEN `famat_inv`(GEN `f`) returns f^{-1} .

GEN `famat_inv_shallow`(GEN `f`) shallow version of `famat_inv`.

GEN `famat_div_shallow`(GEN `f`, GEN `g`) return f/g ; shallow.

GEN `famat_Z_gcd`(GEN `M`, GEN `n`) restrict the *famat* `M` to the prime power dividing `n`.

GEN `to_famat`(GEN `x`, GEN `k`) given an element x and an exponent k , returns the *famat* x^k .

GEN `to_famat_shallow`(GEN `x`, GEN `k`) same, as a shallow function.

Note that it is trivial to break up a *famat* into its two constituent columns: `gel(f,1)` and `gel(f,2)` are the elements and exponents respectively. Conversely, `mkmat2` builds a (shallow) *famat* from two `t_COLS`s of the same length.

The last two functions makes an assumption about the elements: they must be regular algebraic numbers (not *famats*) over a given number field:

GEN `famat_reduce`(GEN `f`) given a *famat* f , returns a *famat* g without repeated elements or 0 exponents, such that the expanded forms of f and g would be equal. Shallow function.

GEN `ZM_famat_limit`(GEN `f`, GEN `limit`) given a *famat* f with `t_INT` entries, returns a *famat* g with all factors larger than `limit` multiplied out as the last entry (with exponent 1).

GEN `famat_to_nf`(GEN `nf`, GEN `f`) You normally never want to do this! This is a simplified form of `nffactorback`, where we do not check the user input for consistency.

The description of `famat_to_nf` says that you do not want to use this function. Then how do we recover genuine number field elements? Well, in most cases, we do not need to: most of the functions useful in this context accept *famats* as inputs, for instance `nfsign`, `nfsign_arch`, `ideallog` and `bnfisunit`. Otherwise, we can generally make good use of a quotient operation (modulo a fixed conductor, modulo ℓ -th powers); see the end of Section 13.1.22.

Caveat. Receiving a *famat* input, `bnfisunit` assumes that it is an algebraic integer, since this is expensive to check, and normally easy to ensure from the user's side; do not feed it ridiculous inputs.

GEN `famat_small_reduce`(GEN `f`) as `famat_reduce`, but for exponents given by a `t_VECSMALL`.

13.1.12 Ideal arithmetic.

Conversion to HNF.

GEN `idealhnf`(GEN `nf`, GEN `x`) returns the HNF of the ideal defined by x : x may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is complicated, but useful to reduce general modules to the canonical form once in a while:

- if strictly less than $N = [K : Q]$ generators are given, x is the \mathbf{Z}_K -module they generate,
- if N or more are given, it is assumed that they form a \mathbf{Z} -basis (that the matrix has maximal rank N). This acts as `mathnf` since the \mathbf{Z}_K -module structure is (taken for granted hence) not taken into account in this case.

Extended ideals are also accepted, their principal part being discarded.

GEN `idealhnf0`(GEN `nf`, GEN `x`, GEN `y`) returns the HNF of the ideal generated by the two algebraic numbers x and y .

The following low-level functions underlie the above two: they all assume that `nf` is a true *nf* and perform no type checks:

GEN `idealhnf_principal`(GEN `nf`, GEN `x`) returns the ideal generated by the algebraic number x .

GEN `idealhnf_shallow`(GEN `nf`, GEN `x`) is `idealhnf` except that the result may not be suitable for `gerepile`: if x is already in HNF, we return x , not a copy!

GEN `idealhnf_two`(GEN `nf`, GEN `v`) assuming $a = v[1]$ is a non-zero `t_INT` and $b = v[2]$ is an algebraic integer, possibly given in regular representation by a `t_MAT` (the multiplication table by b , see `zk_multable`), returns the HNF of $a\mathbf{Z}_K + b\mathbf{Z}_K$.

Operations.

The basic ideal routines accept all `nfs` (`nf`, `bnf`, `bnr`) and ideals in any form, including extended ideals, and return ideals in HNF, or an extended ideal when that makes sense:

GEN `idealadd`(GEN `nf`, GEN `x`, GEN `y`) returns $x + y$.

GEN `idealdiv`(GEN `nf`, GEN `x`, GEN `y`) returns x/y . Returns an extended ideal if x or y is an extended ideal.

GEN `idealmul`(GEN `nf`, GEN `x`, GEN `y`) returns xy . Returns an extended ideal if x or y is an extended ideal.

GEN `idealsqr`(GEN `nf`, GEN `x`) returns x^2 . Returns an extended ideal if x is an extended ideal.

GEN `idealinv`(GEN `nf`, GEN `x`) returns x^{-1} . Returns an extended ideal if x is an extended ideal.

GEN `idealpow`(GEN `nf`, GEN `x`, GEN `n`) returns x^n . Returns an extended ideal if x is an extended ideal.

GEN `idealpows`(GEN `nf`, GEN `ideal`, long `n`) returns x^n . Returns an extended ideal if x is an extended ideal.

GEN `idealmulred`(GEN `nf`, GEN `x`, GEN `y`) returns an extended ideal equal to xy .

GEN `idealpowred`(GEN `nf`, GEN `x`, GEN `n`) returns an extended ideal equal to x^n .

More specialized routines suffer from various restrictions:

GEN `idealdivexact`(GEN `nf`, GEN `x`, GEN `y`) returns x/y , assuming that the quotient is an integral ideal. Much faster than `idealdiv` when the norm of the quotient is small compared to Nx . Strips the principal parts if either x or y is an extended ideal.

GEN `idealdivpowprime`(GEN `nf`, GEN `x`, GEN `pr`, GEN `n`) returns $x\mathfrak{p}^{-n}$, assuming x is an ideal in HNF or a rational number, and `pr` a `prid` attached to \mathfrak{p} . Not suitable for `gerepileupto` since it returns x when $n = 0$.

GEN `idealmulpowprime`(GEN `nf`, GEN `x`, GEN `pr`, GEN `n`) returns $x\mathfrak{p}^n$, assuming x is an ideal in HNF or a rational number, and `pr` a `prid` attached to \mathfrak{p} . Not suitable for `gerepileupto` since it returns x when $n = 0$.

GEN `idealprodprime`(GEN `nf`, GEN `v`) given a list v of prime ideals in `prid` form, return their product. Assume that `nf` is a true `nf` structure.

GEN `idealprod`(GEN `nf`, GEN `v`) given a list v of ideals, return their product.

GEN `idealprodval`(GEN `nf`, GEN `v`, GEN `pr`) given a list v of ideals return the valuation of their product at the prime ideal `pr`.

GEN `idealHNF_mul`(GEN `nf`, GEN `x`, GEN `y`) returns xy , assuming that `nf` is a true nf , x is an integral ideal in HNF and y is an integral ideal in HNF or precompiled form (see below). For maximal speed, the second ideal y may be given in precompiled form $y = [a, b]$, where a is a non-zero `t_INT` and b is an algebraic integer in regular representation (a `t_MAT` giving the multiplication table by the fixed element): very useful when many ideals x are going to be multiplied by the same ideal y . This essentially reduces each ideal multiplication to an $N \times N$ matrix multiplication followed by a $N \times 2N$ modular HNF reduction (modulo $xy \cap \mathbf{Z}$).

GEN `idealHNF_inv`(GEN `nf`, GEN `I`) returns I^{-1} , assuming that `nf` is a true nf and x is a fractional ideal in HNF.

GEN `idealHNF_inv_Z`(GEN `nf`, GEN `I`) returns $(I \cap \mathbf{Z}) \cdot I^{-1}$, assuming that `nf` is a true nf and x is an integral fractional ideal in HNF. The result is an integral ideal in HNF.

Approximation.

GEN `idealaddtoone`(GEN `nf`, GEN `A`, GEN `B`) given two coprime integer ideals A, B , returns $[a, b]$ with $a \in A, b \in B$, such that $a + b = 1$. The result is reduced mod AB , so a, b will be small.

GEN `idealaddtoone_i`(GEN `nf`, GEN `A`, GEN `B`) as `idealaddtoone` except that `nf` must be a true nf , and only a is returned.

GEN `idealaddtoone_raw`(GEN `nf`, GEN `A`, GEN `B`) as `idealaddtoone_i` except that the reduction mod AB is only performed modulo the lcm of $A \cap \mathbf{Z}$ and $B \cap \mathbf{Z}$, which will increase the size of a .

GEN `zkchineseinit`(GEN `nf`, GEN `A`, GEN `B`, GEN `AB`) given two coprime integral ideals A and B (in any form, preferably HNF) and their product AB (in HNF form), initialize a solution to the Chinese remainder problem modulo AB .

GEN `zkchinese`(GEN `zkc`, GEN `x`, GEN `y`) given `zkc` from `zkchineseinit`, and x, y two integral elements given as `t_INT` or `ZC`, return a z modulo AB such that $z = x \pmod A$ and $z = y \pmod B$.

GEN `zkchinese1`(GEN `zkc`, GEN `x`) as `zkchinese` for $y = 1$; useful to lift elements in a nice way from $(\mathbf{Z}_K/A_i)^*$ to $(\mathbf{Z}_K/\prod_i A_i)^*$.

GEN `hnfmerge_get_1`(GEN `A`, GEN `B`) given two square upper HNF integral matrices A, B of the same dimension $n > 0$, return a in the image of A such that $1 - a$ is in the image of B . (By abuse of notation we denote 1 the column vector $[1, 0, \dots, 0]$.) If such an a does not exist, return `NULL`. This is the function underlying `idealaddtoone`.

GEN `idealadddmultoone`(GEN `nf`, GEN `v`) given a list of n (globally) coprime integer ideals $(v[i])$ returns an n -dimensional vector a such that $a[i] \in v[i]$ and $\sum a[i] = 1$. If $[K : \mathbf{Q}] = N$, this routine computes the HNF reduction (with $Gl_{nN}(\mathbf{Z})$ base change) of an $N \times nN$ matrix; so it is well worth pruning "useless" ideals from the list (as long as the ideals remain globally coprime).

GEN `idealapprfact`(GEN `nf`, GEN `fx`) as `idealappr`, except that x must be given in factored form. (This is unchecked.)

GEN `idealcoprime`(GEN `nf`, GEN `x`, GEN `y`). Given 2 integral ideals x and y , returns an algebraic number α such that αx is an integral ideal coprime to y .

GEN `idealcoprimefact`(GEN `nf`, GEN `x`, GEN `fy`) same as `idealcoprime`, except that y is given in factored form, as from `idealfactor`.

GEN `idealchinese`(GEN `nf`, GEN `x`, GEN `y`)

GEN `idealchineseinit`(GEN `nf`, GEN `x`)

13.1.13 Maximal ideals.

The PARI structure attached to maximal ideals is a *prid* (for *prime ideal*), usually produced by `idealprimedec` and `idealfactor`. In this section, we describe the format; other sections will deal with their daily use.

A *prid* attached to a maximal ideal \mathfrak{p} stores the following data: the underlying rational prime p , the ramification degree $e \geq 1$, the residue field degree $f \geq 1$, a p -uniformizer π with valuation 1 at \mathfrak{p} and valuation 0 at all other primes dividing p and a rescaled “anti-uniformizer” τ used to compute valuations. This τ is an algebraic integer such that τ/p has valuation -1 at \mathfrak{p} and is integral at all other primes; in particular, the valuation of $x \in \mathbf{Z}_K$ is positive if and only if the algebraic integer $x\tau$ is divisible by p (easy to check for elements in `t_COL` form).

`GEN pr_get_p(GEN pr)` returns p . Shallow function.

`GEN pr_get_gen(GEN pr)` returns π . Shallow function.

`long pr_get_e(GEN pr)` returns e .

`long pr_get_f(GEN pr)` returns f .

`GEN pr_get_tau(GEN pr)` returns `zk_scalar_or_multable(nf, τ)`, which is the `t_INT` 1 iff p is inert, and a `ZM` otherwise. Shallow function.

`int pr_is_inert(GEN pr)` returns 1 if p is inert, 0 otherwise.

`GEN pr_norm(GEN pr)` returns the norm p^f of the maximal ideal.

`ulong upr_norm(GEN pr)` returns the norm p^f of the maximal ideal, as an `ulong`. Assume that the result does not overflow.

`GEN pr_hnf(GEN pr)` return the HNF of \mathfrak{p} .

`GEN pr_inv(GEN pr)` return the fractional ideal \mathfrak{p}^{-1} , in HNF.

`GEN pr_inv_p(GEN pr)` return the integral ideal $p\mathfrak{p}^{-1}$, in HNF.

`GEN idealprimedec(GEN nf, GEN p)` list of maximal ideals dividing the prime p .

`GEN idealprimedec_limit_f(GEN nf, GEN p, long f)` as `idealprimedec`, limiting the list to primes of residual degree $\leq f$ if f is non-zero.

`GEN idealprimedec_limit_norm(GEN nf, GEN p, GEN B)` as `idealprimedec`, limiting the list to primes of norm $\leq B$, which must be a positive `t_INT`.

`GEN idealprimedec_galois(GEN nf, GEN p)` return a single prime ideal above p .

`GEN idealprimedec_degrees(GEN nf, GEN p)` return a (sorted) `t_VEC` containing the residue degrees $f(\mathfrak{p}/p)$.

`GEN idealprimedec_kummer(GEN nf, GEN Ti, long ei, GEN p)` let nf (true nf) correspond to $K = \mathbf{Q}[X]/(T)$ (T monic `ZX`). Let $T \equiv \prod_i T_i^{e_i} \pmod{p}$ be the factorization of T and let (f, g, h) be as in Dedekind criterion for prime p : $f \equiv \prod T_i$, $g \equiv \prod T_i^{e_i-1}$, $h = (T - fg)/p$, and let D be the gcd of (f, g, h) in $\mathbf{F}_p[X]$. Let T_i (`FpX`) be one irreducible factor T_i not dividing D , with $ei = e_i$. This function returns the prime ideal attached to T_i by Kummer / Dedekind criterion, namely $p\mathbf{Z}_K + T_i(\bar{X})\mathbf{Z}_K$, which has ramification index e_i over p . Shallow function.

`GEN idealHNF_Z_factor(GEN x, GEN *pvN, GEN *pvZ)` given an integral (non-0) ideal x in HNF, compute both the factorization of Nx and of $x \cap \mathbf{Z}$. This returns the vector of prime divisors of both

and sets `*pvN` and `*pvZ` to the corresponding `t_VECSMALL` vector of exponents for the factorization for the Norm and intersection with \mathbf{Z} respectively.

`GEN idealHNF_Z_factor_i(GEN x, GEN fa, GEN *pvN, GEN *pvZ)` internal variant of `idealHNF_Z_factor` where `fa` is either a partial factorization of $x \cap \mathbf{Z}$ ($= x[1, 1]$) or `NULL`. Returns the prime divisors of x above the rational primes in `fa` and attached `vn` and `vZ`. If `fa` is `NULL`, use the full factorization, i.e. identical to `idealHNF_Z_factor`.

`GEN nf_pV_to_prV(GEN)nf, GEN P` given a vector of rational primes P , return the vector of all prime ideals above the $P[i]$.

`GEN nf_deg1_prime(GEN nf)` let `nf` be a true `nf`. This function returns a degree 1 (unramified) prime ideal not dividing `nf.index`. In fact it returns an ideal above the smallest prime $p \geq [K : \mathbf{Q}]$ satisfying those conditions.

`GEN prV_lcm_capZ(GEN L)` given a vector L of `prid` (maximal ideals) return the squarefree positive integer generating their lcm intersected with \mathbf{Z} . Not `gerepile`-safe.

`GEN pr_uniformizer(GEN pr, GEN F)` given a `prid` attached to \mathfrak{p}/p and F in \mathbf{Z} divisible exactly by p , return an F -uniformizer for `pr`, i.e. a t in \mathbf{Z}_K such that $v_{\mathfrak{p}}(t) = 1$ and $(t, F/\mathfrak{p}) = 1$. Not `gerepile`-safe.

13.1.14 Decomposition group.

`GEN idealramfrobenius(GEN nf, GEN gal, GEN pr, GEN ram)` Let K be the number field defined by `nf` and assume K/\mathbf{Q} be a Galois extension with Galois group given `gal=galoisinit(nf)`, and that `pr` is the prime ideal \mathfrak{P} in `prid` format, and that \mathfrak{P} is ramified, and `ram` is its list of ramification groups as output by `idealramgroups`. This function returns a permutation of `gal.group` which defines an automorphism σ in the decomposition group of \mathfrak{P} such that if p is the unique prime number in \mathfrak{P} , then $\sigma(x) \equiv x^p \pmod{\mathbf{P}}$ for all $x \in \mathbf{Z}_K$.

`GEN idealramfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN ram, GEN aut)` as `idealramfrobenius(nf, gal, pr, ram)`.

`GEN idealramgroups_aut(GEN nf, GEN gal, GEN pr, GEN aut)` as `idealramgroups(nf, gal, pr)`.

`GEN idealfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN aut)` faster version of `idealfrobenius(nf, gal, pr)` where `aut` must be equal to `nfgaloispermtobasis(nf, gal)`.

13.1.15 Reducing modulo maximal ideals.

`GEN nfmodprinit(GEN nf, GEN pr)` returns an abstract `modpr` structure, attached to reduction modulo the maximal ideal `pr`, in `idealprimedec` format. From this data we can quickly project any `pr`-integral number field element to the residue field.

`GEN modpr_get_pr(GEN x)` return the `pr` component from a `modpr` structure.

`GEN modpr_get_p(GEN x)` return the p component from a `modpr` structure (underlying rational prime).

`GEN modpr_get_T(GEN x)` return the `T` component from a `modpr` structure: either `NULL` (prime of degree 1) or an irreducible `FpX` defining the residue field over \mathbf{F}_p .

In library mode, it is often easier to use directly

GEN `nf_to_Fq_init`(GEN `nf`, GEN `*ppr`, GEN `*pT`, GEN `*pp`) concrete version of `nfmodprinit`: `nf` and `*ppr` are the inputs, the return value is a `modpr` and `*ppr`, `*pT` and `*pp` are set as side effects.

The input `*ppr` is either a maximal ideal or already a `modpr` (in which case it is replaced by the underlying maximal ideal). The residue field is realized as $\mathbf{F}_p[X]/(T)$ for some monic $T \in \mathbf{F}_p[X]$, and we set `*pT` to T and `*pp` to p . Set $T = \text{NULL}$ if the prime has degree 1 and the residue field is \mathbf{F}_p .

In short, this receives (or initializes) a `modpr` structure, and extracts from it T , p and \mathfrak{p} .

GEN `nf_to_Fq`(GEN `nf`, GEN `x`, GEN `modpr`) returns an `Fq` congruent to x modulo the maximal ideal attached to `modpr`. The output is canonical: all elements in a given residue class are represented by the same `Fq`.

GEN `Fq_to_nf`(GEN `x`, GEN `modpr`) returns an `nf` element lifting the residue field element x , either a `t_INT` or an algebraic integer in `algtobasis` format.

GEN `modpr_genFq`(GEN `modpr`) Returns an `nf` element whose image by `nf_to_Fq` is $X \pmod{T}$, if $\deg T > 1$, else 1.

GEN `zkmodprinit`(GEN `nf`, GEN `pr`) as `nfmodprinit`, but we assume we will only reduce algebraic integers, hence do not initialize data allowing to remove denominators. More precisely, we can in fact still handle an x whose rational denominator is not 0 in the residue field (i.e. if the valuation of x is non-negative at all primes dividing p).

GEN `zk_to_Fq_init`(GEN `nf`, GEN `*pr`, GEN `*T`, GEN `*p`) as `nf_to_Fq_init`, able to reduce only p -integral elements.

GEN `zk_to_Fq`(GEN `x`, GEN `modpr`) as `nf_to_Fq`, for a p -integral x .

GEN `nfM_to_FqM`(GEN `M`, GEN `nf`, GEN `modpr`) reduces a matrix of `nf` elements to the residue field; returns an `FqM`.

GEN `FqM_to_nfM`(GEN `M`, GEN `modpr`) lifts an `FqM` to a matrix of `nf` elements.

GEN `nfV_to_FqV`(GEN `A`, GEN `nf`, GEN `modpr`) reduces a vector of `nf` elements to the residue field; returns an `FqV` with the same type as `A` (`t_VEC` or `t_COL`).

GEN `FqV_to_nfV`(GEN `A`, GEN `modpr`) lifts an `FqV` to a vector of `nf` elements (same type as `A`).

GEN `nfX_to_FqX`(GEN `Q`, GEN `nf`, GEN `modpr`) reduces a polynomial with `nf` coefficients to the residue field; returns an `FqX`.

GEN `FqX_to_nfX`(GEN `Q`, GEN `modpr`) lifts an `FqX` to a polynomial with coefficients in `nf`.

The following functions are technical and avoid computing a true `nfmodpr`:

GEN `pr_basis_perm`(GEN `nf`, GEN `pr`) given a true `nf` structure and a prime ideal `pr` above p , return as a `t_VECSMALL` the $f(\mathfrak{p}/p)$ indices i such that the `nf.zk[i]` mod \mathfrak{p} form an \mathbf{F}_p -basis of the residue field.

GEN `QXQV_to_FpM`(GEN `v`, GEN `T`, GEN `p`) let p be a positive integer, v be a vector of n polynomials with rational coefficients whose denominators are coprime to p , and T be a `ZX` (preferably monic) of degree d whose leading coefficient is coprime to p . Return the $d \times n$ `FpM` whose columns are the $v[i] \pmod{T, p}$ in the canonical basis $1, X, \dots, X^{d-1}$, see `RgX_to_RgC`. This is for instance useful when v contains a \mathbf{Z} -basis of the maximal order of a number field $\mathbf{Q}[X]/(P)$, p is a prime not

dividing the index of P and T is an irreducible factor of $P \bmod p$, attached to a maximal ideal \mathfrak{p} : left-multiplication by the matrix maps number field elements (in basis form) to the residue field of \mathfrak{p} .

13.1.16 Valuations.

`long nfval(GEN nf, GEN x, GEN P)` return $v_P(x)$

Unsafe functions. assume that P, Q are `prid`.

`long ZC_nfval(GEN x, GEN P)` returns $v_P(x)$, assuming x is a `ZC`, representing a non-zero algebraic integer.

`long ZC_nfvalrem(GEN x, GEN P, GEN *newx)` returns $v = v_P(x)$, assuming x is a `ZC`, representing a non-zero algebraic integer, and sets `*newx` to $x\tau^v$ which is an algebraic integer coprime to p .

`int ZC_prdvd(GEN x, GEN P)` returns 1 if P divides x and 0 otherwise. Assumes that x is a `ZC`, representing an algebraic integer. Faster than computing $v_P(x)$.

`int pr_equal(GEN P, GEN Q)` returns 1 if P and Q represent the same maximal ideal: they must lie above the same p and share the same e, f invariants, but the p -uniformizer and τ element may differ. Returns 0 otherwise.

13.1.17 Signatures.

“Signs” of the real embeddings of number field element are represented in additive notation, using the standard identification $(\mathbf{Z}/2\mathbf{Z}, +) \rightarrow (\{-1, 1\}, \times)$, $s \mapsto (-1)^s$.

With respect to a fixed `nf` structure, a selection of real places (a divisor at infinity) is normally given as a `t_VECSMALL` of indices of the roots `nf.roots` of the defining polynomial for the number field. For compatibility reasons, in particular under `GP`, the (obsolete) `vec01` form is also accepted: a `t_VEC` with `gen_0` or `gen_1` entries.

The following internal functions go back and forth between the two representations for the Archimedean part of divisors (`GP`: 0/1 vectors, library: list of indices):

`GEN vec01_to_indices(GEN v)` given a `t_VEC` v with `t_INT` entries return as a `t_VECSMALL` the list of indices i such that $v[i] \neq 0$. (Typically used with 0, 1-vectors but not necessarily so.) If v is already a `t_VECSMALL`, return it: not suitable for `gerepile` in this case.

`GEN vecsmall01_to_indices(GEN v)` as

`vec01_to_indices(zv_to_ZV(v));`

`GEN indices_to_vec01(GEN p, long n)` return the 0/1 vector of length n with ones exactly at the positions $p[1], p[2], \dots$

`GEN nfembed(GEN nf, GEN x, long k)` returns a floating point approximation of the k -th embedding of x (attached to the k -th complex root in `nf.roots`).

`GEN nfsign(GEN nf, GEN x)` x being a number field element and `nf` any form of number field, return the 0 – 1-vector giving the signs of the r_1 real embeddings of x , as a `t_VECSMALL`. Linear algebra functions like `Flv_add_inplace` then allow keeping track of signs in series of multiplications.

If x is a `t_VEC` of number field elements, return the matrix whose columns are the signs of the $x[i]$.

GEN `nfsign_arch`(GEN `nf`, GEN `x`, GEN `arch`) `arch` being a list of distinct real places, either in `vec01` (`t_VEC` with `gen_0` or `gen_1` entries) or `indices` (`t_VECSMALL`) form (see `vec01_to_indices`), returns the signs of x at the corresponding places. This is the low-level function underlying `nfsign`.

int `nfchecksigs`(GEN `nf`, GEN `x`, GEN `pl`) `pl` is a `t_VECSMALL` with r_1 components, all of which are in $\{-1, 0, 1\}$. Return 1 if $\sigma_i(x)pl[i] \geq 0$ for all i , and 0 otherwise.

GEN `nfsign_units`(GEN `bnf`, GEN `archp`, int `add_tu`) `archp` being a divisor at infinity in `indices` form (or `NULL` for the divisor including all real places), return the signs at `archp` of a system of fundamental units for the field, in the same order as `bnf.tufu` if `add_tu` is set; and in the same order as `bnf.fu` otherwise.

GEN `nfsign_from_logarch`(GEN `L`, GEN `invpi`, GEN `archp`) given L the vector of the $\log \sigma(x)$, where σ runs through the (real or complex) embeddings of some number field, `invpi` being a floating point approximation to $1/\pi$, and `archp` being a divisor at infinity in `indices` form, return the signs of x at the corresponding places. This is the low-level function underlying `nfsign_units`; the latter is actually a trivial wrapper `bnf` structures include the $\log \sigma(x)$ for a system of fundamental units of the field.

GEN `set_sign_mod_divisor`(GEN `nf`, GEN `x`, GEN `y`, GEN `sarch`) let $f = f_0 f_\infty$ be a divisor, let `sarch` be the output of `nfarchstar`(`nf`, `f0`, `finf`), let x encode a vector of signs at the places of f_∞ (see below), and let y be a non-zero number field element. Returns z congruent to $y \pmod{f_0}$ (integral if y is) such that z and x have the same signs at f_∞ .

The following formats are supported for x : a $\{0, 1\}$ -vector of signs as a `t_VECSMALL` (0 for positive, 1 for negative); `NULL` for a totally positive element (only 0s); a number field element which is replaced by its signature at f_∞ .

GEN `nfarchstar`(GEN `nf`, GEN `f0`, GEN `finf`) for a divisor $f = f_0 f_\infty$ represented by the integral ideal `f0` in HNF and the `finf` in `indices` form, returns $(\mathbf{Z}_K/f_\infty)^*$ in a form suitable for computations mod f . See `set_sign_mod_divisor`.

GEN `idealprincipalunits`(GEN `nf`, GEN `pr`, long `e`) returns the multiplicative group $(1 + pr)/(1 + pr^e)$ as an abelian group. Faster than `idealstar` when the norm of pr is large, since it avoids (useless) work in the multiplicative group of the residue field.

13.1.18 Maximal order and discriminant, conversion to `nf` structure.

A number field $K = \mathbf{Q}[X]/(T)$ is defined by a monic $T \in \mathbf{Z}[X]$. The low-level function computing a maximal order is

void `nfmaxord`(`nfmaxord_t *S`, GEN `T0`, long `flag`), where the polynomial T_0 is squarefree with integer coefficients. Let K be the étale algebra $\mathbf{Q}[X]/(T_0)$ and let $T = \mathbf{Z}X.Q_normalize(T_0)$, i.e. $T = CT_0(X/L)$ is monic and integral for some $C, Q \in \mathbf{Q}$.

The structure `nfmaxord_t` is initialized by the call; it has the following fields:

```
GEN T0, T, dT, dK; /* T0, T, discriminants of T and K */
GEN unscale; /* the integer L */
GEN index; /* index of power basis in maximal order */
GEN dTP, dTE; /* factorization of |dT|, primes / exponents */
GEN dKP, dKE; /* factorization of |dK|, primes / exponents */
GEN basis; /* Z-basis for maximal order of Q[X]/(T) */
```

The exponent vectors are `t_VECSMALL`. The primes in `dTP` and `dKP` are pseudoprimes, not proven primes. We recommend restricting to $T = T_0$, i.e. either to pass the input polynomial through `ZX_Q_normalize` before the call, or to forget about T_0 and go on with the polynomial T ; otherwise `unscale` $\neq 1$, all data is expressed in terms of $T \neq T_0$, and needs to be converted to T_0 . For instance to convert the basis to $\mathbf{Q}[X]/(T_0)$:

```
RgXV_unscale(S.basis, S.unscale)
```

Instead of passing T (monic `ZX`), one can use the format $[T, listP]$ as in `nfbasis` or `nfinit`, which computes an order which is maximal at a set of primes, but need not be the maximal order.

The `flag` is an or-ed combination of the binary flags, both of them deprecated:

`nf_PARTIALFACT`: do not try to fully factor `dT` and only look for primes less than `primelimit`. In that case, the elements in `dTP` and `dKP` need not all be primes. But the resulting `dK`, `index` and `basis` are correct provided there exists no prime $p > \text{primelimit}$ such that p^2 divides the field discriminant `dK`. This flag is *deprecated*: the $[T, listP]$ format is safer and more flexible.

`nf_ROUND2`: this flag is *deprecated* and now ignored.

`void nfinit_basic(nfmaxord_t *S, GEN T0)` a wrapper around `nfmaxord` (without the deprecated `flag`) that also accepts number field structures (`nf`, `bnf`, ...) for T_0 .

`GEN nfmaxord_to_nf(nfmaxord_t *S, GEN ro, long prec)` convert an `nfmaxord_t` to an `nf` structure at precision `prec`, where `ro` is `NULL`. The argument `ro` may also be set to a vector with $r_1 + r_2$ components containing the roots of $S \rightarrow T$ suitably ordered, i.e. first r_1 `t_REAL` roots, then r_2 `t_COMPLEX` representing the conjugate pairs, but this is *strongly discouraged*: the format is error-prone, and it is hard to compute the roots to the right accuracy in order to achieve `prec` accuracy for the `nf`. This function uses the integer basis $S \rightarrow \text{basis}$ as is, *without* performing LLL-reduction. Unless the basis is already known to be reduced, use rather the following higher-level function:

`GEN nfinit_complete(nfmaxord_t *S, long flag, long prec)` convert an `nfmaxord_t` to an `nf` structure at precision `prec`. The `flag` has the same meaning as in `nfinitall`. If $S \rightarrow \text{basis}$ is known to be reduced, it will be faster to use `nfmaxord_to_nf`.

`GEN indexpartial(GEN T, GEN dT)` T a monic separable `ZX`, `dT` is either `NULL` (no information) or a multiple of the discriminant of T . Let $K = \mathbf{Q}[X]/(T)$ and \mathbf{Z}_K its maximal order. Returns a multiple of the exponent of the quotient group $\mathbf{Z}_K/(\mathbf{Z}[X]/(T))$. In other word, a *denominator* d such that $dx \in \mathbf{Z}[X]/(T)$ for all $x \in \mathbf{Z}_K$.

`GEN FpX_gcd_check(GEN x, GEN y, GEN D)` let x and y be two coprime polynomials with integer coefficients and let D be a factor of the resultant of x and y ; try to factor D by running the Euclidean algorithm on x and y modulo D . This returns `NULL` or a non trivial factor of D . This is the low-level function underlying `poldiscfactors` (applied to x , `ZX_deriv(x)` and the discriminant of x). It succeeds when D has at least two prime divisors p and q such that one sub-resultant of x and y is divisible by p but not by q .

13.1.19 Computing in the class group.

We compute with arbitrary ideal representatives (in any of the various formats seen above), and call

`GEN bnfisprincipal0(GEN bnf, GEN x, long flag)`. The `bnf` structure already contains information about the class group in the form $\bigoplus_{i=1}^n (\mathbf{Z}/d_i\mathbf{Z})g_i$ for canonical integers d_i (with $d_n \mid \dots \mid d_1$ all > 1) and essentially random generators g_i , which are ideals in HNF. We normally do not need the value of the g_i , only that they are fixed once and for all and that any (non-zero) fractional ideal x can be expressed uniquely as $x = (t) \prod_{i=1}^n g_i^{e_i}$, where $0 \leq e_i < d_i$, and (t) is some principal ideal. Computing e is straightforward, but t may be very expensive to obtain explicitly. The routine returns (possibly partial) information about the pair $[e, t]$, depending on `flag`, which is an or-ed combination of the following symbolic flags:

- `nf_GEN` tries to compute t . Returns $[e, t]$, with t an empty vector if the computation failed. This flag is normally useless in non-trivial situations since the next two serve analogous purposes in more efficient ways.

- `nf_GENMAT` tries to compute t in factored form, which is much more efficient than `nf_GEN` if the class group is moderately large; imagine a small ideal $x = (t)g^{10000}$: the norm of t has 10000 as many digits as the norm of g ; do we want to see it as a vector of huge meaningless integers? The idea is to compute e first, which is easy, then compute (t) as $x \prod g_i^{-e_i}$ using successive `idealmulred`, where the ideal reduction extracts small principal ideals along the way, eventually raised to large powers because of the binary exponentiation technique; the point is to keep this principal part in factored *unexpanded* form. Returns $[e, t]$, with t an empty vector if the computation failed; this should be exceedingly rare, unless the initial accuracy to which `bnf` was computed was ridiculously low (and then `bnfinit` should not have succeeded either). Setting/unsetting `nf_GEN` has no effect when this flag is set.

- `nf_GEN_IF_PRINCIPAL` tries to compute t *only* if the ideal is principal ($e = 0$). Returns `gen_0` if the ideal is not principal. Setting/unsetting `nf_GEN` has no effect when this flag is set, but setting/unsetting `nf_GENMAT` is possible.

- `nf_FORCE` in the above, insist on computing t , even if it requires recomputing a `bnf` from scratch. This is a last resort, and normally the accuracy of a `bnf` can be increased without trouble, but it may be that some algebraic information simply cannot be recovered from what we have: see `bnfnewprec`. It should be very rare, though.

In simple cases where you do not care about t , you may use

`GEN isprincipal(GEN bnf, GEN x)`, which is a shortcut for `bnfisprincipal0(bnf, x, 0)`.

The following low-level functions are often more useful:

`GEN isprincipalfact(GEN bnf, GEN C, GEN L, GEN f, long flag)` is about the same as `bnfisprincipal0` applied to $C \prod L[i]^{f[i]}$, where the $L[i]$ are ideals, the $f[i]$ integers and C is either an ideal or `NULL` (omitted). Make sure to include `nf_GENMAT` in `flag`!

`GEN isprincipalfact_or_fail(GEN bnf, GEN C, GEN L, GEN f)` is for delicate cases, where we must be more clever than `nf_FORCE` (it is used when trying to increase the accuracy of a `bnf`, for instance). If performs

```
isprincipalfact(bnf,C, L, f, nf_GENMAT);
```

but if it fails to compute t , it just returns a `t_INT`, which is the estimated precision (in words, as usual) that would have been sufficient to complete the computation. The point is that `nf_FORCE`

does exactly this internally, but goes on increasing the accuracy of the `bnf`, then discarding it, which is a major inefficiency if you intend to compute lots of discrete logs and have selected a precision which is just too low. (It is sometimes not so bad since most of the really expensive data is cached in `bnf` anyway, if all goes well.) With this function, the *caller* may decide to increase the accuracy using `bnfnewprec` (and keep the resulting `bnf!`), or avoid the computation altogether. In any case the decision can be taken at the place where it is most likely to be correct.

`void bnfctestprimes(GEN bnf, GEN B)` is an ingredient to certify unconditionnally a `bnf` computed assuming GRH, cf. `bnfcertify`. Running this function successfully proves that the classes of all prime ideals of norm $\leq B$ belong to the subgroup of the class group generated by the factorbase used to compute the `bnf` (equal to the class group under GRH). If the condition is not true, then (GRH is false and) the function will run forever.

If it is known that primes of norm less than B generate the class group (through variants of Minkowski's convex body or Zimmert's twin classes theorems), then the true class group is proven to be a quotient of `bnf.clgp`.

13.1.20 Floating point embeddings, the T_2 quadratic form.

We assume the `nf` is a true `nf` structure, attached to a number field K of degree n and signature (r_1, r_2) . We saw that

`GEN nf_get_M(GEN nf)` returns the $(r_1 + r_2) \times n$ matrix M giving the embeddings of K , so that if v is an n -th dimensional `t_COL` representing the element $\sum_{i=1}^n v[i]w_i$ of K , then `RgM_RgC_mul(M, v)` represents the embeddings of v . Its first r_1 components are real numbers (`t_INT`, `t_FRAC` or `t_REAL`, usually the latter), and the last r_2 are complex numbers (usually of `t_COMPLEX`, but not necessarily for embeddings of rational numbers).

`GEN embed_T2(GEN x, long r1)` assuming x is the vector of floating point embeddings of some algebraic number v , i.e.

```
x = RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v));
```

returns $T_2(v)$. If the floating point embeddings themselves are not needed, but only the values of T_2 , it is more efficient to restrict to real arithmetic and use

```
gnorml2( RgM_RgC_mul(nf_get_G(nf), algtobasis(nf,v)));
```

`GEN embednorm_T2(GEN x, long r1)` analogous to `embed_T2`, applied to the `gnorm` of the floating point embeddings. Assuming that

```
x = gnorm( RgM_RgC_mul(nf_get_M(nf), algtobasis(nf,v)) );
```

returns $T_2(v)$.

`GEN embed_roots(GEN z, long r1)` given a vector z of $r_1 + r_2$ complex embeddings of the algebraic number v , return the $r_1 + 2r_2$ roots of its characteristic polynomial. Shallow function.

`GEN embed_disc(GEN z, long r1, long prec)` given a vector z of $r_1 + r_2$ complex embeddings of the algebraic number v , return a floating point approximation of the discriminant of its characteristic polynomial as a `t_REAL` of precision `prec`.

`GEN embed_norm(GEN x, long r1)` given a vector z of $r_1 + r_2$ complex embeddings of the algebraic number v , return (a floating point approximation of) the norm of v .

13.1.21 Ideal reduction, low level.

In the following routines *nf* is a true **nf**, attached to a number field K of degree n :

GEN `nf_get_Gtwist(GEN nf, GEN v)` assuming v is a **t_VECSMALL** with $r_1 + r_2$ entries, let

$$\|x\|_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i} \varepsilon_i |\sigma_i(x)|^2,$$

where as usual the σ_i are the (real and) complex embeddings and $\varepsilon_i = 1$, resp. 2, for a real, resp. complex place. This is a twisted variant of the T_2 quadratic form, the standard Euclidean form on $K \otimes \mathbf{R}$. In applications, only the relative size of the v_i will matter.

Let $G_v \in M_n(\mathbf{R})$ be a square matrix such that if $x \in K$ is represented by the column vector X in terms of the fixed **Z**-basis of \mathbf{Z}_K in *nf*, then

$$\|x\|_v^2 = {}^t(G_v X) \cdot G_v X.$$

(This is a kind of Cholesky decomposition.) This function returns a rescaled copy of G_v , rounded to nearest integers, specifically `RM_round_maxrank(G_v)`. Suitable for `gerepileupto`, but does not collect garbage. For convenience, also allow $v = \text{NULL}$ (`nf_get_roundG`) and v a **t_MAT** as output from the function itself: in both these cases, shallow function.

GEN `nf_get_Gtwist1(GEN nf, long i)`. Simple special case. Returns the twisted G matrix attached to the vector v whose entries are all 0 except the i -th one, which is equal to 10.

GEN `idealpseudomin(GEN x, GEN G)`. Let x, G be two **ZMs**, such that the product Gx is well-defined. This returns a “small” integral linear combinations of the columns of x , given by the LLL-algorithm applied to the lattice Gx . Suitable for `gerepileupto`, but does not collect garbage.

In applications, x is an integral ideal, G approximates a Cholesky form for the T_2 quadratic form as returned by `nf_get_Gtwist`, and we return a small element a in the lattice (x, T_2) . This is used to implement `idealred`.

GEN `idealpseudomin_nonscalar(GEN x, GEN G)`. As `idealpseudomin`, but we insist of returning a non-scalar a (`ZV_isscalar` is false), if the dimension of x is > 1 .

In the interpretation where x defines an integral ideal on a fixed \mathbf{Z}_K basis whose first element is 1, this means that a is not rational.

GEN `idealpseudored(GEN x, GEN G)`. As `idealpseudomin` but we return the full reduced **Z**-basis of x as a **t_MAT** instead of a single vector.

GEN `idealred_elt(GEN nf, GEN x)` shortcut for

```
idealpseudomin(x, nf_get_roundG(nf))
```

13.1.22 Ideal reduction, high level.

Given an ideal x this means finding a “simpler” ideal in the same ideal class. The public GP function is of course available

`GEN idealred0(GEN nf, GEN x, GEN v)` finds an $a \in K^*$ such that $(a)x$ is integral of small norm and returns it, as an ideal in HNF. What “small” means depends on the parameter v , see the GP description. More precisely, a is returned by `idealpseudomin((xZ)x(-1), G)` divided by $x_{\mathbf{Z}}$, where $x_{\mathbf{Z}} = (x \cap \mathbf{Z})$ and where G is `nf_get_Gtwist(nf, v)` for $v \neq \text{NULL}$ and `nf_get_roundG(nf)` otherwise.

Usually one sets $v = \text{NULL}$ to obtain an element of small T_2 norm in x :

`GEN idealred(GEN nf, GEN x)` is a shortcut for `idealred0(nf, x, NULL)`.

The function `idealred` remains complicated to use: in order not to lose information x must be an extended ideal, otherwise the value of a is lost. There is a subtlety here: the principal ideal (a) is easy to recover, but a itself is an instance of the principal ideal problem which is very difficult given only an nf (once a bnf structure is available, `bnfisprincipal0` will recover it).

`GEN idealmoddivisor(GEN bnr, GEN x)` A proof-of-concept implementation, useless in practice. If `bnr` is attached to some modulus f , returns a “small” ideal in the same class as x in the ray class group modulo f . The reason why this is useless is that using extended ideals with principal part in a computation, there is a simple way to reduce them: simply reduce the generator of the principal part in $(\mathbf{Z}_K/f)^*$.

`GEN famat_to_nf_moddivisor(GEN nf, GEN g, GEN e, GEN bid)` given a true nf attached to a number field K , a bid structure attached to a modulus f , and an algebraic number in factored form $\prod g[i]^{e[i]}$, such that $(g[i], f) = 1$ for all i , returns a small element in \mathbf{Z}_K congruent to it mod f . Note that if f contains places at infinity, this includes sign conditions at the specified places.

A simpler case when the conductor has no place at infinity:

`GEN famat_to_nf_modideal_coprime(GEN nf, GEN g, GEN e, GEN f, GEN expo)` as above except that the ideal f is now integral in HNF (no need for a full bid), and we pass the exponent of the group $(\mathbf{Z}_K/f)^*$ as `expo`; any multiple will also do, at the expense of efficiency. Of course if a bid for f is available, it is easy to extract f and the exact value of `expo` from it (the latter is the first elementary divisor in the group structure). A useful trick: if you set `expo` to *any* positive integer, the result is correct up to `expo`-th powers, hence exact if `expo` is a multiple of the exponent; this is useful when trying to decide whether an element is a square in a residue field for instance! (take `expo=2`).

`GEN nf_to_Fp_coprime(GEN nf, GEN x, GEN modpr)` this low-level function is variant of `famat_to_nf_modideal_coprime`: nf is a true nf structure, `modpr` is from `zkmodprinit` attached to a prime of degree 1 above the prime number p , and x is either a number field element or a `famat` factorization matrix. We finally assume that no component of x has a denominator p .

What to do when the $g[i]$ are not coprime to f , but only $\prod g[i]^{e[i]}$ is? Then the situation is more complicated, and we advise to solve it one prime divisor of f at a time. Let v the valuation attached to a maximal ideal `pr` and assume $v(f) = k > 0$:

`GEN famat_makecoprime(GEN nf, GEN g, GEN e, GEN pr, GEN prk, GEN expo)` returns an element in $(\mathbf{Z}_K/\text{pr}^k)^*$ congruent to the product $\prod g[i]^{e[i]}$, assumed to be globally coprime to f . As above, `expo` is any positive multiple of the exponent of $(\mathbf{Z}_K/\text{pr}^k)^*$, for instance $(Nv - 1)p^{k-1}$,

if p is the underlying rational prime. You may use other values of `expo` (see the useful trick in `famat_to_nf_modideal_coprime`).

`GEN Idealstarprk(GEN nf, GEN pr, long k, long flag)` same as `Idealstar` for $I = \mathfrak{p}^k$

13.1.23 Class field theory.

Under GP, a class-field theoretic description of a number field is given by a triple A, B, C , where the defining set $[A, B, C]$ can have any of the following forms: $[bnr]$, $[bnr, subgroup]$, $[bnf, modulus]$, $[bnf, modulus, subgroup]$. You can still use directly all of (`libpari`'s routines implementing) GP's functions as described in Chapter 3, but they are often awkward in the context of `libpari` programming. In particular, it does not make much sense to always input a triple A, B, C because of the fringe $[bnf, modulus, subgroup]$. The first routine to call, is thus

`GEN Buchray(GEN bnf, GEN mod, long flag)` initializes a `bnr` structure from `bnf` and modulus `mod`. `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (if omitted, do not return a `bnr`, only the ray class group as an abelian group). In fact, a single value of `flag` actually makes sense: `nf_GEN | nf_INIT` to initialize a proper `bnr`: removing `nf_GEN` saves very little time, but the corresponding crippled `bnr` structure will raise errors in most class field theoretic functions. Possibly also 0 to quickly compute the ray class group structure; `bnrclassno` is faster if we only need the *order* of the ray class group.

Now we have a proper `bnr` encoding a `bnf` and a modulus, we no longer need the $[bnf, modulus]$ and $[bnf, modulus, subgroup]$ forms, which would internally call `Buchray` anyway. Recall that a subgroup H is given by a matrix in HNF, whose column express generators of H on the fixed generators of the ray class group that stored in our `bnr`. You may also code the trivial subgroup by `NULL`.

`GEN bnrconductor(GEN bnr, GEN H, long flag)` see the documentation of the GP function.

`GEN bnrconductor_i(GEN bnr, GEN H, long flag)` shallow variant of `bnrconductor`. Useful when `flag = 2` and the conductor is the `bnr` modulus: avoids copying the `bnr` (wasteful).

`long bnriscconductor(GEN bnr, GEN H)` returns 1 if the class field defined by the subgroup H (of the ray class group mod f coded in `bnr`) has conductor f . Returns 0 otherwise.

`GEN bnrchar_primitive(GEN bnr, GEN chi, GEN bnrc)` Given a normalized character `chi = [d, c]` on `bnr.clgp` (see `char_normalize`) of conductor `bnrc.mod`, compute the primitive character `chic` on `bnrc.clgp` equivalent to `chi`, given as a normalized character $[D, C]$: `chic(bnrc.gen[i])` is $\zeta_D^{C[i]}$, where D is minimal. It is easier to use `bnrconductor_i(bnr, chi, 2)`, but the latter recomputes `bnrc` for each new character.

`GEN bnrdisc(GEN bnr, GEN H, long flag)` returns the discriminant and signature of the class field defined by `bnr` and H . See the description of the GP function for details. `flag` is an or-ed combination of the flags `rnf_REL` (output relative data) and `rnf_COND` (return 0 unless the modulus is the conductor).

`GEN bnr surjection(GEN BNR, GEN bnr)` `BNR` and `bnr` defined over the same field K , for moduli F and f with $F | f$, returns the matrix of the canonical surjection $\text{Cl}_K(F) \rightarrow \text{Cl}_K(f)$ (giving the image of the fixed ray class group generators of `BNR` in terms of the ones in `bnr`).

`GEN ABC_to_bnr(GEN A, GEN B, GEN C, GEN *H, int addgen)` This is a quick conversion function designed to go from the too general (inefficient) A, B, C form to the preferred `bnr, H` form for class fields. Given A, B, C as explained above (omitted entries coded by `NULL`), return the attached `bnr`, and set H to the attached subgroup. If `addgen` is 1, make sure that if the `bnr` needed to be computed, then it contains generators.

13.1.24 Grunwald–Wang theorem.

`GEN nfgwkummer(GEN nf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `nf` contains suitable roots of unity, and directly using Kummer theory to construct the extension.

`GEN bnfgwgeneric(GEN bnf, GEN Lpr, GEN Ld, GEN pl, long var)` low-level version of `nfgrunwaldwang`, assuming that `bnf` is a `bnfinit` structure, and calling `rnfkummer` to construct the extension.

13.1.25 Relative equations, Galois conjugates.

`GEN nfissquarefree(GEN nf, GEN P)` given P a polynomial with coefficients in nf , return 1 if P is squarefree, and 0 otherwise. If is allowed (though less efficient) to replace nf by a monic `ZX` defining the field.

`GEN rnfequationall(GEN A, GEN B, long *pk, GEN *pLPRS)` A is either an nf type (corresponding to a number field K) or an irreducible `ZX` defining a number field K . B is an irreducible polynomial in $K[X]$. Returns an absolute equation C (over \mathbf{Q}) for the number field $K[X]/(B)$. C is the characteristic polynomial of $b + ka$ for some roots a of A and b of B , and k is a small rational integer. Set `*pk` to k .

If `pLPRS` is not `NULL` set it to $[h_0, h_1]$, $h_i \in \mathbf{Q}[X]$, where $h_0 + h_1Y$ is the last non-constant polynomial in the pseudo-Euclidean remainder sequence attached to $A(Y)$ and $B(X - kY)$, leading to $C = \text{Res}_Y(A(Y), B(X - kY))$. In particular $a := -h_0/h_1$ is a root of A in $\mathbf{Q}[X]/(C)$, and $X - ka$ is a root of B .

`GEN nf_rnfeq(GEN A, GEN B)` wrapper around `rnfequationall` to allow mapping $K \rightarrow L$ (`eltup`) and converting elements of L between absolute and relative form (`reltoabs`, `abstorel`), *without* computing a full rnf structure, which is useful if the relative integral basis is not required. In fact, since A may be a `t_POL` or an nf , the integral basis of the base field is not needed either. The return value is the same as `rnf_get_map`. Shallow function.

`GEN nf_rnfeqsimple(GEN nf, GEN relpol)` as `nf_rnfeq` except some fields are omitted, so that only the `abstorel` operation is supported. Shallow function.

`GEN eltabstorel(GEN rnfeq, GEN x)` `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltabstorel` is more robust), `nf_rnfeq` or `nf_rnfeqsimple`, return x as an element of L/K , i.e. as a `t_POLMOD` with `t_POLMOD` coefficients. Shallow function.

`GEN eltabstorel_lift(GEN rnfeq, GEN x)` same as `eltabstorel`, except that x is returned in partially lifted form, i.e. as a `t_POL` with `t_POLMOD` coefficients.

`GEN eltretoabs(GEN rnfeq, GEN x)` `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltreloabs` is more robust) or `nf_rnfeq`, return x in absolute form.

`GEN nf_nfzk(GEN nf, GEN rnfeq)` `rnfeq` as given by `nf_rnfeq`, `nf` a true nf structure, return a suitable representation of `nf.zk` allowing quick computation of the map $K \rightarrow L$ by the function `nfeltup`, *without* computing a full rnf structure, which is useful if the relative integral basis is not required. The computed value is the same as in `rnf_get_nfzk`. Shallow function.

`GEN nfeltup(GEN nf, GEN x, GEN zknf)` `zknf` and is initialized by `nf_nfzk` or `rnf_get_nfzk` (but in this case `rnfeltup` is more robust); `nf` is a true nf structure for K , returns $x \in K$ as a (lifted) element of L , in absolute form.

GEN `rnfdisc_factored(GEN nf, GEN pol, GEN *pd)` variant of `rnfdisc` returning the relative discriminant ideal *factorization*, and setting `*pd` to the discriminant as an element in $K^*/(K^*)^2$. Shallow function.

GEN `Rg_nffix(const char *f, GEN T, GEN c, int lift)` given a ZX T and a “coefficient” c supposedly belonging to $\mathbf{Q}[y]/(T)$, check whether this is the case and return a cleaned up version of c . The string f is the calling function name, used to report errors.

This means that c must be one of `t_INT`, `t_FRAC`, `t_POL` in the variable y with rational coefficients, or `t_POLMOD` modulo T which lift to a rational `t_POL` as above. The cleanup consists in the following improvements:

- `t_POL` coefficients are reduced modulo T .
- `t_POL` and `t_POLMOD` belonging to \mathbf{Q} are converted to rationals, `t_INT` or `t_FRAC`.
- if `lift` is non-zero, convert `t_POLMOD` to `t_POL`, and otherwise convert `t_POL` to `t_POLMODs` modulo T .

GEN `RgX_nffix(const char *f, GEN T, GEN P, int lift)` check whether P is a polynomial with coefficients in the number field defined by the absolute equation $T(y) = 0$, where T is a ZX and returns a cleaned up version of P . This checks whether P is indeed a `t_POL` with variable compatible with coefficients in $\mathbf{Q}[y]/(T)$, i.e.

$$\text{varncmp}(\text{varn}(P), \text{varn}(T)) < 0$$

and applies `Rg_nffix` to each coefficient.

GEN `RgV_nffix(const char *f, GEN T, GEN P, int lift)` as `RgX_nffix` for a vector of coefficients.

GEN `polmod_nffix(const char *f, GEN rnf, GEN x, int lift)` given a `t_POLMOD` x supposedly defining an element of *rnf*, check this and perform `Rg_nffix` cleanups.

GEN `polmod_nffix2(const char *f, GEN T, GEN P, GEN x, int lift)` as in `polmod_nffix`, where the relative extension is explicitly defined as $L = (\mathbf{Q}[y]/(T))[x]/(P)$, instead of by an *rnf* structure.

long `numberofconjugates(GEN T, long pinit)` returns a quick multiple for the number of \mathbf{Q} -automorphism of the (integral, monic) `t_POL` T , from modular factorizations, starting from prime `pinit` (you can set it to 2). This upper bounds often coincides with the actual number of conjugates. Of course, you should use `nfgaloisconj` to be sure.

GEN `nfroots_if_split(GEN *pt, GEN T)` let `*pt` point either to a number field structure or an irreducible ZX, defining a number field K . Given T a monic squarefree polynomial with coefficients in \mathbf{Z}_K , return the list of roots of `pol` in K if the polynomial splits completely, and `NULL` otherwise. In other words, this checks whether $K[X]/(T)$ is normal over K (hence Galois since T is separable by assumption).

In the case where `*pT` is a ZX, the function has to compute internally a conditional `nf` attached to K , whose `nf.zk` may not define the maximal order \mathbf{Z}_K (see `nfroots`); `*pT` is then replaced by the conditional `nf` to avoid losing that information.

13.1.26 Cyclotomics units.

GEN `nfrootsof1(GEN nf)` returns a two-component vector $[w, z]$ where w is the number of roots of unity in the number field nf , and z is a primitive w -th root of unity.

GEN `nfcyclotomicunits(GEN nf, GEN zu)` where `zu` is as output by `nfrootsof1(nf)`, return the vector of the cyclotomic units in `nf` expressed over the integral basis.

13.1.27 Obsolete routines.

Still provided for backward compatibility, but should not be used in new programs. They will eventually disappear.

GEN `zidealstar(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN)`

GEN `zidealstarinit(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_INIT)`

GEN `zidealstarinitgen(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN|nf_INIT)`

GEN `buchimag(GEN D, GEN c1, GEN c2, GEN gCO)` short for

`Buchquad(D,gtodouble(c1),gtodouble(c2), /*ignored*/0)`

GEN `buchreal(GEN D, GEN gsens, GEN c1, GEN c2, GEN RELSUP, long prec)` short for

`Buchquad(D,gtodouble(c1),gtodouble(c2), prec)`

The following use a naming scheme which is error-prone and not easily extensible; besides, they compute generators as per `nf_GEN` and not `nf_GENMAT`. Don't use them:

GEN `isprincipalforce(GEN bnf, GEN x)`

GEN `isprincipalgen(GEN bnf, GEN x)`

GEN `isprincipalgenforce(GEN bnf, GEN x)`

GEN `isprincipalraygen(GEN bnr, GEN x)`, use `bnrisprincipal`.

Variants on `polred`: use `polredbest`.

GEN `factoredpolred(GEN x, GEN fa)`

GEN `factoredpolred2(GEN x, GEN fa)`

GEN `smallpolred(GEN x)`

GEN `smallpolred2(GEN x)`, use `Polred`.

GEN `polred0(GEN x, long flag, GEN p)`

GEN `polredabs(GEN x)`

GEN `polredabs2(GEN x)`

GEN `polredabsall(GEN x, long flun)`

Superseded by `bnrdiscclist0`:

GEN `discrayabslist(GEN bnf, GEN listes)`

GEN `discrayabslistarch(GEN bnf, GEN arch, long bound)`

Superseded by `idealappr` (*flagis* ignored)

GEN `idealappr0(GEN nf, GEN x, long flag)`

13.2 Galois extensions of \mathbb{Q} .

This section describes the data structure output by the function `galoisinit`. This will be called a `gal` structure in the following.

13.2.1 Extracting info from a `gal` structure.

The functions below expect a `gal` structure and are shallow. See the documentation of `galoisinit` for the meaning of the member functions.

`GEN gal_get_pol(GEN gal)` returns `gal.pol`

`GEN gal_get_p(GEN gal)` returns `gal.p`

`GEN gal_get_e(GEN gal)` returns the integer e such that `gal.mod==gal.pe`.

`GEN gal_get_mod(GEN gal)` returns `gal.mod`.

`GEN gal_get_roots(GEN gal)` returns `gal.roots`.

`GEN gal_get_invvdm(GEN gal)` `gal[4]`.

`GEN gal_get_den(GEN gal)` return `gal[5]`.

`GEN gal_get_group(GEN gal)` returns `gal.group`.

`GEN gal_get_gen(GEN gal)` returns `gal.gen`.

`GEN gal_get_orders(GEN gal)` returns `gal.orders`.

13.2.2 Miscellaneous functions.

`GEN nfgaloispermtobasis(GEN nf, GEN gal)` return the images of the field generator by the automorphisms `gal.orders` expressed on the integral basis `nf.zk`.

`GEN nfgaloismatrix(GEN nf, GEN s)` returns the ZM attached to the automorphism s , seen as a linear operator expressed on the number field integer basis. This allows to use

```
M = nfgaloismatrix(nf, s);
sx = ZM_ZC_mul(M, x); /* or RgM_RgC_mul(M, x) if x is not integral */
```

instead of

```
sx = nfgaloisapply(nf, s, x);
```

for an algebraic integer x .

13.3 Quadratic number fields and quadratic forms.

13.3.1 Checks.

`void check_quaddisc(GEN x, long *s, long *mod4, const char *f)` checks whether the GEN x is a quadratic discriminant (`t_INT`, not a square, congruent to 0, 1 modulo 4), and raise an exception otherwise. Set `*s` to the sign of x and `*mod4` to x modulo 4 (0 or 1).

`void check_quaddisc_real(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is positive.

`void check_quaddisc_imag(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is negative.

13.3.2 Class number.

The function `quadclassunit` uses index calculus and runs in subexponential time but it assumes the truth of the GRH. For imaginary quadratic orders, it is comparatively slow for *small* values, say $|D| \leq 10^{18}$. Here are some alternatives:

`GEN classno(GEN D)` corresponds to `qfbclassno(D,0)` and is only useful for $D < 0$, uses a baby-step giant-step technique and runs in time $O(D^{1/4})$. The result is guaranteed correct for $|D| < 2 \cdot 10^{10}$ and fastest in that range. For larger values of $|D|$, the algorithm is no longer rigorous and may give incorrect results (we know no concrete example); it also becomes relatively less interesting compared to `quadclassunit`.

`GEN classno2(GEN D)` corresponds to `qfbclassno(D,1)` and runs in time $O(D^{1/2})$; it is provided for testing purposes only: it is never competitive.

`GEN hclassno(GEN d)` returns the Hurwitz-Kronecker class number $H(d)$. These play a central role in trace formulas and are usually needed for many consecutive values of d . Thus, the function uses a cache so that later calls for *small* consecutive values of d are instantaneous, see `getcache`. Large values of d ($d > 500000$) call `quadclassunit` individually and are not memoized.

`GEN hclassno6(GEN d)` assuming $d > 0$, returns the integer $6H(d)$. This is a low-level function behind `hclassno`.

`ulong hclassno6u(ulong d)` assuming $d > 0$, returns the integer $6H(d)$.

13.3.3 `t_QFI`, `t_QFR`.

`GEN qfi(GEN x, GEN y, GEN z)` creates the `t_QFI` (x, y, z) .

`GEN qfr(GEN x, GEN y, GEN z, GEN d)` creates the `t_QFR` (x, y, z) with distance component d .

`GEN qfr_1(GEN q)` given a `t_QFR` q , return the unit form q^0 .

`GEN qfi_1(GEN q)` given a `t_QFI` q , return the unit form q^0 .

`int qfb_equal1(GEN q)` returns 1 if the `t_QFI` or `t_QFR` q is the unit form.

13.3.3.1 Composition.

GEN `qficomp`(GEN `x`, GEN `y`) compose the two `t_QFI` x and y , then reduce the result. This is the same as `gmul(x,y)`.

GEN `qfrcomp`(GEN `x`, GEN `y`) compose the two `t_QFR` x and y , then reduce the result. This is the same as `gmul(x,y)`.

GEN `qfisqr`(GEN `x`) as `qficomp(x,y)`.

GEN `qfrsqr`(GEN `x`) as `qfrcomp(x,y)`.

Same as above, *without* reducing the result:

GEN `qficompraw`(GEN `x`, GEN `y`)

GEN `qfrcompraw`(GEN `x`, GEN `y`)

GEN `qfisqrraw`(GEN `x`)

GEN `qfrsqrraw`(GEN `x`)

GEN `qfbcompraw`(GEN `x`, GEN `y`) compose two `t_QFIs` or two `t_QFRs`, without reduce the result.

13.3.3.2 Powering.

GEN `powgi`(GEN `x`, GEN `n`) computes x^n (will work for many more types than `t_QFI` and `t_QFR`, of course). Reduce the result.

GEN `qfrpow`(GEN `x`, GEN `n`) computes x^n for a `t_QFR` x , reducing along the way. If the distance component is initially 0, leave it alone; otherwise update it.

GEN `qfbpowraw`(GEN `x`, long `n`) compute x^n (pure composition, no reduction), for a `t_QFI` or `t_QFR` x .

GEN `qfipowraw`(GEN `x`, long `n`) as `qfbpowraw`, for a `t_QFI` x .

GEN `qfrpowraw`(GEN `x`, long `n`) as `qfbpowraw`, for a `t_QFR` x .

13.3.3.3 Order, discrete log.

GEN `qfi_order`(GEN `q`, GEN `o`) assuming that the `t_QFI` q has order dividing o , compute its order in the class group. The order can be given in all formats allowed by generic discrete log functions, the preferred format being [`ord`, `fa`] (`t_INT` and its factorization).

GEN `qfi_log`(GEN `a`, GEN `g`, GEN `o`) given a `t_QFI` a and assuming that the `t_QFI` g has order o , compute an integer k such that $a^k = g$. Return `cgetg(1, t_VEC)` if there are no solutions. Uses a generic Pollig-Hellman algorithm, then either Shanks (small o) or Pollard rho (large o) method. The order can be given in all formats allowed by generic discrete log functions, the preferred format being [`ord`, `fa`] (`t_INT` and its factorization).

GEN `qfi_Shanks`(GEN `a`, GEN `g`, long `n`) given a `t_QFI` a and assuming that the `t_QFI` g has (small) order n , compute an integer k such that $a^k = g$. Return `cgetg(1, t_VEC)` if there are no solutions. Directly uses Shanks algorithm, which is inefficient when n is composite.

13.3.3.4 Solve, Cornacchia.

The following functions underly `qfbsolve`; p denotes a prime number.

`GEN qfisolvp(GEN Q, GEN p)` solves $Q(x, y) = p$ over the integers, for a `t_QFI` Q . Return `gen_0` if there are no solutions.

`GEN qfrsolvp(GEN Q, GEN p)` solves $Q(x, y) = p$ over the integers, for a `t_QFR` Q . Return `gen_0` if there are no solutions.

`long cornacchia(GEN d, GEN p, GEN *px, GEN *py)` solves $x^2 + dy^2 = p$ over the integers, where $d > 0$. Return 1 if there is a solution (and store it in `*x` and `*y`), 0 otherwise.

`long cornacchia2(GEN d, GEN p, GEN *px, GEN *py)` as `cornacchia`, for the equation $x^2 + dy^2 = 4p$.

`long cornacchia2_sqrt(GEN d, GEN p, GEN b, GEN *px, GEN *py)` as `cornacchia2`, where $p > 2$ and b is the smallest squareroot of d modulo p .

13.3.3.5 Prime forms.

`GEN primeform_u(GEN x, ulong p) t_QFI` whose first coefficient is the prime p .

`GEN primeform(GEN x, GEN p, long prec)`

13.3.4 Efficient real quadratic forms. Unfortunately, `t_QFRs` are very inefficient, and are only provided for backward compatibility.

- they do not contain needed quantities, which are thus constantly recomputed (the discriminant D , \sqrt{D} and its integer part),

- the distance component is stored in logarithmic form, which involves computing one extra logarithm per operation. It is much more efficient to store its exponential, computed from ordinary multiplications and divisions (taking exponent overflow into account), and compute its logarithm at the very end.

Internally, we have two representations for real quadratic forms:

- `qfr3`, a container $[a, b, c]$ with at least 3 entries: the three coefficients; the idea is to ignore the distance component.

- `qfr5`, a container with at least 5 entries $[a, b, c, e, d]$: the three coefficients a `t_REAL` d and a `t_INT` e coding the distance component $2^{Ne}d$, in exponential form, for some large fixed N .

It is a feature that `qfr3` and `qfr5` have no specified length or type. It implies that a `qfr5` or `t_QFR` will do whenever a `qfr3` is expected. Routines using these objects all require a global context, provided by a `struct qfr_data *`:

```
struct qfr_data {
    GEN D;          /* discriminant, t_INT */
    GEN sqrtD;     /* sqrt(D), t_REAL */
    GEN isqrtD;    /* floor(sqrt(D)), t_INT */
};
```

`void qfr_data_init(GEN D, long prec, struct qfr_data *S)` given a discriminant $D > 0$, initialize S for computations at precision `prec` (\sqrt{D} is computed to that initial accuracy).

All functions below are shallow, and not stack clean.

GEN `qfr3_comp`(GEN `x`, GEN `y`, struct `qfr_data *S`) compose two `qfr3`, reducing the result.

GEN `qfr3_pow`(GEN `x`, GEN `n`, struct `qfr_data *S`) compute x^n , reducing along the way.

GEN `qfr3_red`(GEN `x`, struct `qfr_data *S`) reduce x .

GEN `qfr3_rho`(GEN `x`, struct `qfr_data *S`) perform one reduction step; `qfr3_red` just performs reduction steps until we hit a reduced form.

GEN `qfr3_to_qfr`(GEN `x`, GEN `d`) recover an ordinary `t_QFR` from the `qfr3` x , adding distance component d .

Before we explain `qfr5`, recall that it corresponds to an ideal, that reduction corresponds to multiplying by a principal ideal, and that the distance component is a clever way to keep track of these principal ideals. More precisely, reduction consists in a number of reduction steps, going from the form (a, b, c) to $\rho(a, b, c) = (c, -b \bmod 2c, *)$; the distance component is multiplied by (a floating point approximation to) $(b + \sqrt{D})/(b - \sqrt{D})$.

GEN `qfr5_comp`(GEN `x`, GEN `y`, struct `qfr_data *S`) compose two `qfr5`, reducing the result, and updating the distance component.

GEN `qfr5_pow`(GEN `x`, GEN `n`, struct `qfr_data *S`) compute x^n , reducing along the way.

GEN `qfr5_red`(GEN `x`, struct `qfr_data *S`) reduce x .

GEN `qfr5_rho`(GEN `x`, struct `qfr_data *S`) perform one reduction step.

GEN `qfr5_dist`(GEN `e`, GEN `d`, long `prec`) decode the distance component from exponential (`qfr5`-specific) to logarithmic form (as in a `t_QFR`).

GEN `qfr_to_qfr5`(GEN `x`, long `prec`) convert a `t_QFR` to a `qfr5` with initial trivial distance component (= 1).

GEN `qfr5_to_qfr`(GEN `x`, GEN `d`), assume x is a `qfr5` and d was the original distance component of some `t_QFR` that we converted using `qfr_to_qfr5` to perform efficiently a number of operations. Convert x to a `t_QFR` with the correct (logarithmic) distance component.

13.4 Linear algebra over \mathbf{Z} .

13.4.1 Hermite and Smith Normal Forms.

GEN `ZM_hnf`(GEN `x`) returns the upper triangular Hermite Normal Form of the ZM x (removing 0 columns), using the `ZM_hnfall` algorithm. If you want the true HNF, use `ZM_hnfall(x, NULL, 0)`.

GEN `ZM_hnfmod`(GEN `x`, GEN `d`) returns the HNF of the ZM x (removing 0 columns), assuming the `t_INT` d is a multiple of the determinant of x . This is usually faster than `ZM_hnf` (and uses less memory) if the dimension is large, > 50 say.

GEN `ZM_hnfmodid`(GEN `x`, GEN `d`) returns the HNF of the matrix $(x \mid d\text{Id})$ (removing 0 columns), for a ZM x and a `t_INT` d .

GEN `ZM_hnfmodprime`(GEN `x`, GEN `p`) returns the HNF of the matrix $(x \mid p\text{Id})$ (removing 0 columns), for a ZM x and a prime number p . The algorithm involves only \mathbf{F}_p -linear algebra and is faster than `ZM_hnfmodid` (which will call it when d is prime).

GEN `ZM_hnfmodall`(GEN `x`, GEN `d`, long `flag`) low-level function underlying the `ZM_hnfmod` variants. If `flag` is 0, calls `ZM_hnfmod(x,d)`; `flag` is an or-ed combination of:

- `hnf_MODID` call `ZM_hnfmodid` instead of `ZM_hnfmod`,
- `hnf_PART` return as soon as we obtain an upper triangular matrix, saving time. The pivots are non-negative and give the diagonal of the true HNF, but the entries to the right of the pivots need not be reduced, i.e. they may be large or negative.
- `hnf_CENTER` returns the centered HNF, where the entries to the right of a pivot p are centered residues in $[-p/2, p/2[$, hence smallest possible in absolute value, but possibly negative.

GEN `ZM_hnfmodall_i`(GEN `x`, GEN `d`, long `flag`) as `ZM_hnfmodall` without final garbage collection. Not `gerepile-safe`.

GEN `ZM_hnfall`(GEN `x`, GEN `*U`, long `remove`) returns the upper triangular HNF H of the ZM x ; if U is not NULL, set it to the matrix U such that $xU = H$. If `remove` = 0, H is the true HNF, including 0 columns; if `remove` = 1, delete the 0 columns from H but do not update U accordingly (so that the integer kernel may still be recovered): we no longer have $xU = H$; if `remove` = 2, remove 0 columns from H and update U so that $xU = H$. The matrix U is square and invertible unless `remove` = 2.

This routine uses a naive algorithm which is potentially exponential in the dimension (due to coefficient explosion) but is fast in practice, although it may require lots of memory. The base change matrix U may be very large, when the kernel is large.

GEN `ZM_hnfall_i`(GEN `x`, GEN `*U`, long `remove`) as `ZM_hnfall` without final garbage collection. Not `gerepile-safe`.

GEN `ZM_hnfperm`(GEN A , GEN `*ptU`, GEN `*ptperm`) returns the hnf $H = PAU$ of the matrix PA , where P is a suitable permutation matrix, and $U \in \text{Gl}_n(\mathbf{Z})$. P is chosen so as to (heuristically) minimize the size of U ; in this respect it is less efficient than `ZM_hnf111` but usually faster. Set `*ptU` to U and `*ptperm` to a `t_VECSMALL` representing the row permutation attached to $P = (\delta_{i, \text{perm}[i]}$. If `ptU` is set to NULL, U is not computed, saving some time; although useless, setting `ptperm` to NULL is also allowed.

GEN `ZM_hnf_knapsack`(GEN `x`) given a ZM x , compute its HNF h . Return h if it has the knapsack property: every column contains only zeroes and ones and each row contains a single 1; return NULL otherwise. Not suitable for `gerepile`.

GEN `ZM_hnf111`(GEN `x`, GEN `*U`, int `remove`) returns the HNF H of the ZM x ; if U is not NULL, set it to the matrix U such that $xU = H$. The meaning of `remove` is the same as in `ZM_hnfall`.

This routine uses the LLL variant of Havas, Majewski and Mathews, which is polynomial time, but rather slow in practice because it uses an exact LLL over the integers instead of a floating point variant; it uses polynomial space but lots of memory is needed for large dimensions, say larger than 300. On the other hand, the base change matrix U is essentially optimally small with respect to the L_2 norm.

GEN `ZM_hnfcenter`(GEN M). Given a ZM in HNF M , update it in place so that non-diagonal entries belong to a system of *centered* residues. Not suitable for `gerepile`.

Some direct applications: the following routines apply to upper triangular integral matrices; in practice, these come from HNF algorithms.

GEN `hnf_divscale`(GEN A, GEN B, GEN t) A an upper triangular ZM, B a ZM, t an integer, such that $C := tA^{-1}B$ is integral. Return C .

GEN `hnf_invscale`(GEN A, GEN t) A an upper triangular ZM, t an integer such that $C := tA^{-1}$ is integral. Return C . Special case of `hnf_divscale` when B is the identity matrix.

GEN `hnf_solve`(GEN A, GEN B) A a ZM in upper HNF (not necessarily square), B a ZM or ZC. Return $A^{-1}B$ if it is integral, and NULL if it is not.

GEN `hnf_invimage`(GEN A, GEN b) A a ZM in upper HNF (not necessarily square), b a ZC. Return $A^{-1}b$ if it is integral, and NULL if it is not.

int `hnfdivide`(GEN A, GEN B) A and B are two upper triangular ZM. Return 1 if $A^{-1}B$ is integral, and 0 otherwise.

Smith Normal Form.

GEN `ZM_snf`(GEN x) returns the Smith Normal Form (vector of elementary divisors) of the ZM x .

GEN `ZM_snfall`(GEN x, GEN *U, GEN *V) returns `ZM_snf`(x) and sets U and V to unimodular matrices such that $UxV = D$ (diagonal matrix of elementary divisors). Either (or both) U or V may be NULL in which case the corresponding matrix is not computed.

GEN `ZV_snfall`(GEN d, GEN *U, GEN *V) here d is a ZV; same as `ZM_snfall` applied to `diagonal`(d), but faster.

GEN `ZM_snfall_i`(GEN x, GEN *U, GEN *V, int returnvec) same as `ZM_snfall`, except that, depending on the value of `returnvec`, we either return a diagonal matrix (as in `ZM_snfall`, `returnvec` is 0) or a vector of elementary divisors (as in `ZM_snf`, `returnvec` is 1).

void `ZM_snfclean`(GEN d, GEN U, GEN V) assuming d, U, V come from $d = \text{ZM_snfall}(x, \&U, \&V)$, where U or V may be NULL, cleans up the output in place. This means that elementary divisors equal to 1 are deleted and U, V are updated. The output is not suitable for `gerepileupto`.

void `ZV_snf_trunc`(GEN D) given a vector D of elementary divisors (i.e. a ZV such that $d_i \mid d_{i+1}$), truncate it *in place* to leave out the trivial divisors (equal to 1).

GEN `ZM_snf_group`(GEN H, GEN *U, GEN *Uinv) this function computes data to go back and forth between an abelian group (of finite type) given by generators and relations, and its canonical SNF form. Given an abstract abelian group with generators $g = (g_1, \dots, g_n)$ and a vector $X = (x_i) \in \mathbf{Z}^n$, we write gX for the group element $\sum_i x_i g_i$; analogously if M is an $n \times r$ integer matrix gM is a vector containing r group elements. The group neutral element is 0; by abuse of notation, we still write 0 for a vector of group elements all equal to the neutral element. The input is a full relation matrix H among the generators, i.e. a ZM (not necessarily square) such that $gX = 0$ for some $X \in \mathbf{Z}^n$ if and only if X is in the integer image of H , so that the abelian group is isomorphic to $\mathbf{Z}^n / \text{Im}H$. *The routine assumes that H is in HNF*; replace it by its HNF if it is not the case. (Of course this defines the same group.)

Let G a minimal system of generators in SNF for our abstract group: if the d_i are the elementary divisors ($\dots \mid d_2 \mid d_1$), each G_i has either infinite order ($d_i = 0$) or order $d_i > 1$. Let D the matrix with diagonal (d_i) , then

$$GD = 0, \quad G = gU_{\text{inv}}, \quad g = GU,$$

for some integer matrices U and U_{inv} . Note that these are not even square in general; even if square, there is no guarantee that these are unimodular: they are chosen to have minimal entries given the known relations in the group and only satisfy $D \mid (UU_{\text{inv}} - \text{Id})$ and $H \mid (U_{\text{inv}}U - \text{Id})$.

The function returns the vector of elementary divisors (d_i); if U is not NULL, it is set to U ; if U_{inv} is not NULL it is set to U_{inv} . The function is not memory clean.

GEN `ZV_snf_group`(GEN d , GEN $*\text{new}U$, GEN $*\text{new}U_i$), here d is a ZV; same as `ZM_snf_group` applied to `diagonal(d)`, but faster.

The following routines underly the various `matrixqz` variants. In all case the $m \times n$ `t_MAT` x is assumed to have rational (`t_INT` and `t_FRAC`) coefficients

GEN `QM_ImQ_hnf`(GEN x) returns an HNF basis for $\text{Im}_{\mathbf{Q}}x \cap \mathbf{Z}^n$.

GEN `QM_ImZ_hnf`(GEN x) returns an HNF basis for $\text{Im}_{\mathbf{Z}}x \cap \mathbf{Z}^n$.

GEN `QM_ImQ_hnfall`(GEN A , GEN $*pB$, long `remove`) as `QM_ImQ_hnf`, further returning the transformation matrix as in `ZM_hnfall`.

GEN `QM_ImZ_hnfall`(GEN A , GEN $*pB$, long `remove`) as `QM_ImZ_hnf`, further returning the transformation matrix as in `ZM_hnfall`.

GEN `QM_minors_coprime`(GEN x , GEN D), assumes $m \geq n$, and returns a matrix in $M_{m,n}(\mathbf{Z})$ with the same \mathbf{Q} -image as x , such that the GCD of all $n \times n$ minors is coprime to D ; if D is NULL, we want the GCD to be 1.

The following routines are simple wrappers around the above ones and are normally useless in library mode:

GEN `hnf`(GEN x) checks whether x is a ZM, then calls `ZM_hnf`. Normally useless in library mode.

GEN `hnfmod`(GEN x , GEN d) checks whether x is a ZM, then calls `ZM_hnfmod`. Normally useless in library mode.

GEN `hnfmodid`(GEN x , GEN d) checks whether x is a ZM, then calls `ZM_hnfmodid`. Normally useless in library mode.

GEN `hnfall`(GEN x) calls `ZM_hnfall`(x , $\&U$, 1) and returns $[H, U]$. Normally useless in library mode.

GEN `hnflll`(GEN x) calls `ZM_hnflll`(x , $\&U$, 1) and returns $[H, U]$. Normally useless in library mode.

GEN `hnfperm`(GEN x) calls `ZM_hnfperm`(x , $\&U$, $\&P$) and returns $[H, U, P]$. Normally useless in library mode.

GEN `smith`(GEN x) checks whether x is a ZM, then calls `ZM_snf`. Normally useless in library mode.

GEN `smithall`(GEN x) checks whether x is a ZM, then calls `ZM_snfall`(x , $\&U$, $\&V$) and returns $[U, V, D]$. Normally useless in library mode.

Some related functions over $K[X]$, K a field:

GEN `gsmith`(GEN A) the input matrix must be square, returns the elementary divisors.

GEN `gsmithall`(GEN A) the input matrix must be square, returns the $[U, V, D]$, D diagonal, such that $UAV = D$.

GEN `RgM_hnfall`(GEN A , GEN $*pB$, long `remove`) analogous to `ZM_hnfall`.

GEN `smithclean`(GEN z) cleanup the output of `smithall` or `gsmithall` (delete elementary divisors equal to 1, updating base change matrices).

13.4.2 The LLL algorithm.

The basic GP functions and their immediate variants are normally not very useful in library mode. We briefly list them here for completeness, see the documentation of `qflll` and `qflllgram` for details:

- GEN `qflll0(GEN x, long flag)`

GEN `lll(GEN x) flag= 0`

GEN `lllint(GEN x) flag= 1`

GEN `lllkerim(GEN x) flag= 4`

GEN `lllkeringen(GEN x) flag= 5`

GEN `lllgen(GEN x) flag= 8`

- GEN `qflllgram0(GEN x, long flag)`

GEN `lllgram(GEN x) flag= 0`

GEN `lllgramint(GEN x) flag= 1`

GEN `lllgramkerim(GEN x) flag= 4`

GEN `lllgramkeringen(GEN x) flag= 5`

GEN `lllgramgen(GEN x) flag= 8`

The basic workhorse underlying all integral and floating point LLLs is

GEN `ZM_lll(GEN x, double D, long flag)`, where x is a ZM; $D \in]1/4, 1[$ is the Lovász constant determining the frequency of swaps during the algorithm: a larger values means better guarantees for the basis (in principle smaller basis vectors) but longer running times (suggested value: $D = 0.99$).

Important. This function does not collect garbage and its output is not suitable for either `gerepile` or `gerepileupto`. We expect the caller to do something simple with the output (e.g. matrix multiplication), then collect garbage immediately.

`flag` is an or-ed combination of the following flags:

- `LLL_GRAM`. If set, the input matrix x is the Gram matrix ${}^t v v$ of some lattice vectors v .
- `LLL_INPLACE`. If unset, we return the base change matrix U , otherwise the transformed matrix xU or ${}^t U x U$ (`LLL_GRAM`). Implies `LLL_IM` (see below).
- `LLL_KEEP_FIRST`. The first vector in the output basis is the same one as was originally input. Provided this is a shortest non-zero vector of the lattice, the output basis is still LLL-reduced. This is used to reduce maximal orders of number fields with respect to the T_2 quadratic form, to ensure that the first vector in the output basis corresponds to 1 (which is a shortest vector).
- `LLL_COMPATIBLE`. This is a no-op on 64-bit kernels; on 32-bit kernels, restrict to 64-bit-compatible accuracies in the course of LLL algorithms. This is very likely to produce identical results on all kernels, but this is not guaranteed.

The last three flags are mutually exclusive, either 0 or a single one must be set:

- `LLL_KER` If set, only return a kernel basis K (not LLL-reduced).

- `LLL_IM` If set, only return an LLL-reduced lattice basis T . (This is implied by `LLL_INPLACE`).
- `LLL_ALL` If set, returns a 2-component vector $[K, T]$ corresponding to both kernel and image.

`GEN lllfp(GEN x, double D, long flag)` is a variant for matrices with inexact entries: x is a matrix with real coefficients (types `t_INT`, `t_FRAC` and `t_REAL`), D and $flag$ are as in `ZM_111`. The matrix is rescaled, rounded to nearest integers, then fed to `ZM_111`. The flag `LLL_INPLACE` is still accepted but less useful (it returns an LLL-reduced basis attached to rounded input, instead of an exact base change matrix).

`GEN ZM_111_norms(GEN x, double D, long flag, GEN *ptB)` slightly more general version of `ZM_111`, setting `*ptB` to a vector containing the squared norms of the Gram-Schmidt vectors (b_i^*) attached to the output basis (b_i), $b_i^* = b_i + \sum_{j < i} \mu_{i,j} b_j^*$.

`GEN lllintpartial_inplace(GEN x)` given a `ZM x` of maximal rank, returns a partially reduced basis (b_i) for the space spanned by the columns of x : $|b_i \pm b_j| \geq |b_i|$ for any two distinct basis vectors b_i, b_j . This is faster than the LLL algorithm, but produces much larger bases.

`GEN lllintpartial(GEN x)` as `lllintpartial_inplace`, but returns the base change matrix U from the canonical basis to the b_i , i.e. xU is the output of `lllintpartial_inplace`.

`GEN RM_round_maxrank(GEN G)` given a matrix G with real floating point entries and independent columns, let G_e be the rescaled matrix $2^e G$ rounded to nearest integers, for $e \geq 0$. Finds a small e such that the rank of G_e is equal to the rank of G (its number of columns) and return G_e . This is useful as a preconditioning step to speed up LLL reductions, see `nf_get_Gtwist`. Suitable for `gerepileupto`, but does not collect garbage.

13.4.3 Linear dependencies.

The following functions underly the `lindep` GP function:

`GEN lindep(GEN v)` real/complex entries, guess that about only the 80% leading bits of the input are correct.

`GEN lindep_bit(GEN v, long b)` real/complex entries, explicit form of the above: multiply the input by 2^b and round to nearest integer before looking for a linear dependency. Truncating dubious bits allows to find better relations.

`GEN lindepfull_bit(GEN v, long b)` as `lindep_bit` but return a matrix M with $n = \#v$ columns and r rows, with $r = n + 1$ (if v is real) or $n + 2$ (general case) which is an LLL-reduced basis of the lattice formed by concatenating vertically an identity matrix and the floor of $2^b \text{real}(v)$ and $2^b \text{imag}(v)$ if $r = n + 2$. The first n rows of M potentially correspond to relations: whenever the last $r - n$ entries of a column are small. The function `lindep_bit` essentially returns the first column of M truncated to n components.

`GEN lindep_padic(GEN v)` p -adic entries.

`GEN lindep_Xadic(GEN v)` polynomial entries.

`GEN deplin(GEN v)` returns a non-zero kernel vector for a `t_MAT` input.

Deprecated routine:

`GEN lindep2(GEN x, long dig)` analogous to `lindep_bit`, with `dig` counting decimal digits.

13.4.4 Reduction modulo matrices.

GEN `ZC_hnfremdiv`(GEN `x`, GEN `y`, GEN `*Q`) assuming y is an invertible ZM in HNF and x is a ZC, returns the ZC R equal to $x \bmod y$ (whose i -th entry belongs to $[-y_{i,i}/2, y_{i,i}/2[$). Stack clean unless x is already reduced (in which case, returns x itself, not a copy). If Q is not NULL, set it to the ZC such that $x = yQ + R$.

GEN `ZM_hnfdivrem`(GEN `x`, GEN `y`, GEN `*Q`) reduce each column of the ZM x using `ZC_hnfremdiv`. If Q is not NULL, set it to the ZM such that $x = yQ + R$.

GEN `ZC_hnfrem`(GEN `x`, GEN `y`) alias for `ZC_hnfremdiv(x,y,NULL)`.

GEN `ZM_hnfrem`(GEN `x`, GEN `y`) alias for `ZM_hnfremdiv(x,y,NULL)`.

GEN `ZC_reducemodmatrix`(GEN `v`, GEN `y`) Let y be a ZM, not necessarily square, which is assumed to be LLL-reduced (otherwise, very poor reduction is expected). Size-reduces the ZC v modulo the \mathbf{Z} -module Y spanned by y : if the columns of y are denoted by (y_1, \dots, y_{n-1}) , we return $y_n \equiv v$ modulo Y , such that the Gram-Schmidt coefficients $\mu_{n,j}$ are less than $1/2$ in absolute value for all $j < n$. In short, y_n is almost orthogonal to Y .

GEN `ZM_reducemodmatrix`(GEN `v`, GEN `y`) Let y be as in `ZC_reducemodmatrix`, and v be a ZM. This returns a matrix v which is congruent to v modulo the \mathbf{Z} -module spanned by y , whose columns are size-reduced. This is faster than repeatedly calling `ZC_reducemodmatrix` on the columns since most of the Gram-Schmidt coefficients can be reused.

GEN `ZC_reducemodlll`(GEN `v`, GEN `y`) Let y be an arbitrary ZM, LLL-reduce it then call `ZC_reducemodmatrix`.

GEN `ZM_reducemodlll`(GEN `v`, GEN `y`) Let y be an arbitrary ZM, LLL-reduce it then call `ZM_reducemodmatrix`.

Besides the above functions, which were specific to integral input, we also have:

GEN `reducemodinvertible`(GEN `x`, GEN `y`) y is an invertible matrix and x a `t_COL` or `t_MAT` of compatible dimension. Returns $x - y[y^{-1}x]$, which has small entries and differs from x by an integral linear combination of the columns of y . Suitable for `gerepileupto`, but does not collect garbage.

GEN `closemodinvertible`(GEN `x`, GEN `y`) returns $x - \text{reducemodinvertible}(x, y)$, i.e. an integral linear combination of the columns of y , which is close to x .

GEN `reducemodlll`(GEN `x`, GEN `y`) LLL-reduce the non-singular ZM y and call `reducemodinvertible` to find a small representative of $x \bmod y\mathbf{Z}^n$. Suitable for `gerepileupto`, but does not collect garbage.

13.5 Finite abelian groups and characters.

13.5.1 Abstract groups.

A finite abelian group G in GP format is given by its Smith Normal Form as a pair $[h, d]$ or triple $[h, d, g]$. Here h is the cardinality of G , (d_i) is the vector of elementary divisors, and (g_i) is a vector of generators. In short, $G = \oplus_{i \leq n} (\mathbf{Z}/d_i \mathbf{Z}) g_i$, with $d_n \mid \dots \mid d_2 \mid d_1$ and $\prod d_i = h$.

Let $e(x) := \exp(2i\pi x)$. For ease of exposition, we restrict to complex-valued characters, but everything applies to more general fields K where e denotes a morphism $(\mathbf{Q}, +) \rightarrow (K^*, \times)$ such that $e(a/b)$ denotes a b -th root of unity.

A *character* on the abelian group $\oplus (\mathbf{Z}/d_j \mathbf{Z}) g_j$ is given by a row vector $\chi = [a_1, \dots, a_n]$ such that $\chi(\prod g_j^{n_j}) = e(\sum a_j n_j / d_j)$.

`GEN cyc_normalize(GEN d)` shallow function. Given a vector $(d_i)_{i \leq n}$ of elementary divisors for a finite group (no d_i vanish), returns the vector $D = [1]$ if $n = 0$ (trivial group) and $[d_1, d_1/d_2, \dots, d_1/d_n]$ otherwise. This will allow to define characters as $\chi(\prod g_j^{x_j}) = e(\sum_j x_j a_j D_j / D_1)$, see `char_normalize`.

`GEN char_normalize(GEN chi, GEN ncyc)` shallow function. Given a character $\text{chi} = (a_j)$ and ncyc from `cyc_normalize` above, returns the normalized representation $[d, (n_j)]$, such that $\chi(\prod g_j^{x_j}) = \zeta_d^{\sum_j n_j x_j}$, where $\zeta_d = e(1/d)$ and d is *minimal*. In particular, d is the order of chi . Shallow function.

`GEN char_simplify(GEN D, GEN N)` given a quasi-normalized character $[D, (N_j)]$ such that $\chi(\prod g_j^{x_j}) = \zeta_D^{\sum_j N_j x_j}$, but where we only assume that D is a multiple of the character order, return a normalized character $[d, (n_j)]$ with d *minimal*. Shallow function.

`GEN char_denormalize(GEN cyc, GEN d, GEN n)` given a normalized representation $[d, n]$ (where d need not be minimal) of a character on the abelian group with abelian divisors `cyc`, return the attached character (where the image of each generator g_i is given in terms of roots of unity of different orders `cyc[i]`).

`GEN charconj(GEN cyc, GEN chi)` return the complex conjugate of chi .

`GEN charmul(GEN cyc, GEN a, GEN b)` return the product character $a \times b$.

`GEN chardiv(GEN cyc, GEN a, GEN b)` returns the character $a/b = a \times \bar{b}$.

`int char_check(GEN cyc, GEN chi)` return 1 if chi is a character compatible with cyclic factors `cyc`, and 0 otherwise.

`GEN cyc2elts(GEN d)` given a `t_VEC` $d = (d_1, \dots, d_n)$ of non-negative integers, return the vector of all `t_VECSMALLS` of length n whose i -th entry lies in $[0, d_i]$. Assumes that the product of the d_i fits in a `long`.

13.5.2 Dirichlet characters.

The functions in this section are specific to characters on $(\mathbf{Z}/N\mathbf{Z})^*$. The argument G is a special `bid` structure as returned by `znstar0(N, nf_INIT)`. In this case, there are additional ways to input character via Conrey's representation. The character `chi` is either a `t_INT` (Conrey label), a `t_COL` (a Conrey logarithm) or a `t_VEC` (generic character on `bid.gen` as explained in the previous subsection). The following low-level functions are called by GP's generic character functions.

`int zncharcheck(GEN G, GEN chi)` return 1 if `chi` is a valid character and 0 otherwise.

`GEN zncharconj(GEN G, GEN chi)` as `charconj`.

`GEN znchardiv(GEN G, GEN a, GEN b)` as `chardiv`.

`GEN zncharker(GEN G, GEN chi)` as `charker`.

`GEN znchareval(GEN G, GEN chi, GEN n, GEN z)` as `chareval`.

`GEN zncharmul(GEN G, GEN a, GEN b)` as `charmul`.

`GEN zncharpow(GEN G, GEN a, GEN n)` as `charpow`.

`GEN zncharorder(GEN G, GEN chi)` as `charorder`.

The following functions handle characters in Conrey notation (attached to Conrey generators, not `G.gen`):

`int znconrey_check(GEN cyc, GEN chi)` return 1 if `chi` is a valid Conrey logarithm and 0 otherwise.

`GEN znconrey_normalized(GEN G, GEN chi)` return normalized character attached to `chi`, as in `char_normalize` but on Conrey generators.

`GEN znconreyfromchar(GEN G, GEN chi)` return Conrey logarithm attached to the generic (`t_VEC`, on `G.gen`)

`GEN znconreyfromchar_normalized(GEN G, GEN chi)` return normalized Conrey character attached to the generic (`t_VEC`, on `G.gen`) character `chi`.

`GEN znconreylog_normalize(GEN G, GEN m)` given a Conrey logarithm m (`t_COL`), return the attached normalized Conrey character, as in `char_normalize` but on Conrey generators.

`GEN znchar_quad(GEN G, GEN D)` given a non-zero `t_INT` D congruent to 0, 1 mod 4, return $(D/.)$ as a character modulo N , given by a Conrey logarithm (`t_COL`). Assume that $|D|$ divides N .

`GEN Zideallog(GEN G, GEN x)` return the `znconreylog` of x expressed on `G.gen`, i.e. the ordinary discrete logarithm from `ideallog`.

`GEN ncharvecexpo(GEN G, GEN nchi)` given $nchi = [d, n]$ a quasi-normalized character (d may be a multiple of the character order), i.e. $\chi(g_i) = e(n[i]/d)$ for all Conrey or SNF generators g_i (as usual, we use SNF generators if n is a `t_VEC` and the Conrey generators otherwise). Return a `t_VECSMALL` v such that $v[i] = -1$ if $(i, N) > 1$ else $\chi(i) = e(v[i]/d)$, $1 \leq i \leq N$.

13.6 Central simple algebras.

13.6.1 Initialization.

Low-level routines underlying `alginit`.

`GEN alg_csa_table(GEN nf, GEN mt, long v, long maxord)` algebra defined by a multiplication table.

`GEN alg_cyclic(GEN rnf, GEN aut, GEN b, long maxord)` cyclic algebra $(L/K, \sigma, b)$.

`GEN alg_hasse(GEN nf, long d, GEN hi, GEN hf, long v, long maxord)` algebra defined by local Hasse invariants.

`GEN alg_hilbert(GEN nf, GEN a, GEN b, long v, long maxord)` quaternion algebra.

`GEN alg_matrix(GEN nf, long n, long v, GEN L, long maxord)` matrix algebra.

`GEN alg_complete(GEN rnf, GEN aut, GEN hi, GEN hf, long maxord)` cyclic algebra $(L/K, \sigma, b)$ with b computed from the Hasse invariants.

13.6.2 Type checks.

`void checkalg(GEN a)` raise an exception if a was not initialized by `alginit`.

`void checklat(GEN al, GEN lat)` raise an exception if `lat` is not a valid full lattice in the algebra `al`.

`void checkhasse(GEN nf, GEN hi, GEN hf, long n)` raise an exception if (hi, hf) do not describe valid Hasse invariants of a central simple algebra of degree n over `nf`.

`long alg_type(GEN al)` internal function called by `algtype`: assume `al` was created by `alginit` (thereby saving a call to `checkalg`). Return values are symbolic rather than numeric:

- `al_NULL`: not a valid algebra.
- `al_TABLE`: table algebra output by `alhtableinit`.
- `al_CSA`: central simple algebra output by `alginit` and represented by a multiplication table over its center.
- `al_CYCLIC`: central simple algebra output by `alginit` and represented by a cyclic algebra.

`long alg_model(GEN al, GEN x)` given an element x in algebra `al`, check for inconsistencies (raise a type error) and return the representation model used for x :

- `al_ALGEBRAIC`: `basistoalg` form, algebraic representation.
- `al_BASIS`: `algtobasis` form, column vector on the integral basis.
- `al_MATRIX`: matrix with coefficients in an algebra.
- `al_TRIVIAL`: trivial algebra of degree 1; can be understood as both basis or algebraic form (since $e_1 = 1$).

13.6.3 Shallow accessors.

All these routines assume their argument was initialized by `alginit` and provide minor speedups compared to the GP equivalent. The routines returning a GEN are shallow.

`long alg_get_absdim(GEN al)` low-level version of `algabsdim`.

`long alg_get_dim(GEN al)` low-level version of `algdim`.

`long alg_get_degree(GEN al)` low-level version of `algdegree`.

`GEN alg_get_aut(GEN al)` low-level version of `algaut`.

`GEN alg_get_auts(GEN al)`, given a cyclic algebra $al = (L/K, \sigma, b)$ of degree n , returns the vector of σ^i , $1 \leq i < n$.

`GEN alg_get_b(GEN al)` low-level version of `algb`.

`GEN alg_get_basis(GEN al)` low-level version of `algbasis`.

`GEN alg_get_center(GEN al)` low-level version of `algcenter`.

`GEN alg_get_char(GEN al)` low-level version of `algchar`.

`GEN alg_get_hasse_f(GEN al)` low-level version of `alghassef`.

`GEN alg_get_hasse_i(GEN al)` low-level version of `alghassei`.

`GEN alg_get_invbasis(GEN al)` low-level version of `alginvbasis`.

`GEN alg_get_multable(GEN al)` low-level version of `algmultable`.

`GEN alg_get_relmultable(GEN al)` low-level version of `algrelmultable`.

`GEN alg_get_splittingfield(GEN al)` low-level version of `algsplittingfield`.

`GEN alg_get_abssplitting(GEN al)` returns the absolute *mf* structure attached to the *mf* returned by `algsplittingfield`.

`GEN alg_get_splitpol(GEN al)` returns the relative polynomial defining the *mf* returned by `algsplittingfield`.

`GEN alg_get_splittingdata(GEN al)` low-level version of `algsplittingdata`.

`GEN alg_get_splittingbasis(GEN al)` the matrix *Lbas* from `algsplittingdata`

`GEN alg_get_splittingbasisinv(GEN al)` the matrix *Lbasinv* from `algsplittingdata`.

`GEN alg_get_tracebasis(GEN al)` returns the traces of the basis elements; used by `algtrace`.

`GEN alglat_get_primbasis(GEN lat)` from the description of *lat* as λL with $L \subset \mathcal{O}_0$ and $\lambda \in \mathbf{Q}$, returns a basis of *L*.

`GEN alglat_get_scalar(GEN lat)` from the description of *lat* as λL with $L \subset \mathcal{O}_0$ and $\lambda \in \mathbf{Q}$, returns λ .

13.6.4 Other low-level functions.

`GEN conjclasses_algcenter(GEN cc, GEN p)` low-level function underlying `alggroupcenter`, where *cc* is the output of `groupelts_to_conjclasses`, and *p* is either NULL or a prime number. Not stack clean.

`GEN algsimpledec_ss(GEN al, long maps)` assuming that *al* is semisimple, returns the second component of `algsimpledec(al, maps)`.

Chapter 14: Elliptic curves and arithmetic geometry

This chapter is quite short, but is added as a placeholder, since we expect the library to expand in that direction.

14.1 Elliptic curves.

Elliptic curves are represented in the Weierstrass model

$$(E) : y^2z + a_1xyz + a_3yz = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

by the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$. Points in the projective plane are represented as follows: the point at infinity $(0 : 1 : 0)$ is coded as $[0]$, a finite point $(x : y : 1)$ outside the projective line at infinity $z = 0$ is coded as $[x, y]$. Note that other points at infinity than $(0 : 1 : 0)$ cannot be represented; this is harmless, since they do not belong to any of the elliptic curves E above.

Points on the curve are just projective points as described above, they are not tied to a curve in any way: the same point may be used in conjunction with different curves, provided it satisfies their equations (if it does not, the result is usually undefined). In particular, the point at infinity belongs to all elliptic curves.

As with `factor` for polynomial factorization, the 5-tuple $[a_1, a_2, a_3, a_4, a_6]$ implicitly defines a base ring over which the curve is defined. Point coordinates must be operation-compatible with this base ring (`gadd`, `gmul`, `gdiv` involving them should not give errors).

14.1.1 Types of elliptic curves.

We call a 5-tuple as above an `ell5`; most functions require an `ell` structure, as returned by `ellinit`, which contains additional data (usually dynamically computed as needed), depending on the base field.

`GEN ellinit(GEN E, GEN D, long prec)`, returns an `ell` structure, attached to the elliptic curve E : either an `ell5`, a pair $[a_4, a_6]$ or a `t_STR` in Cremona's notation, e.g. "11a1". The optional D (`NULL` to omit) describes the domain over which the curve is defined.

14.1.2 Type checking.

`void checkell(GEN e)` raise an error unless e is an `ell`.

`int checkell_i(GEN e)` return 1 if e is an `ell` and 0 otherwise.

`void checkell5(GEN e)` raise an error unless e is an `ell` or an `ell5`.

`void checkellpt(GEN z)` raise an error unless z is a point (either finite or at infinity).

`long ell_get_type(GEN e)` returns the domain type over which the curve is defined, one of

`t_ELL_Q` the field of rational numbers;

`t_ELL_NF` a number field;

`t_ELL_Qp` the field of p -adic numbers, for some prime p ;

`t_ELL_Fp` a prime finite field, base field elements are represented as Fp , i.e. a `t_INT` reduced modulo p ;

`t_ELL_Fq` a non-prime finite field (a prime finite field can also be represented by this subtype, but this is inefficient), base field elements are represented as `t_FFELT`;

`t_ELL_Rg` none of the above.

`void checkell_Fq(GEN e)` checks whether e is an `ell`, defined over a finite field (either prime or non-prime). Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Q(GEN e)` checks whether e is an `ell`, defined over \mathbf{Q} . Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Qp(GEN e)` checks whether e is an `ell`, defined over some \mathbf{Q}_p . Otherwise the function raises a `pari_err_TYPE` exception.

`void checkellisog(GEN v)` raise an error unless v is an isogeny, from `ellisogeny`.

14.1.3 Extracting info from an `ell` structure.

These functions expect an `ell` argument. If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

14.1.3.1 All domains.

`GEN ell_get_a1(GEN e)`

`GEN ell_get_a2(GEN e)`

`GEN ell_get_a3(GEN e)`

`GEN ell_get_a4(GEN e)`

`GEN ell_get_a6(GEN e)`

`GEN ell_get_b2(GEN e)`

`GEN ell_get_b4(GEN e)`

`GEN ell_get_b6(GEN e)`

`GEN ell_get_b8(GEN e)`

`GEN ell_get_c4(GEN e)`

`GEN ell_get_c6(GEN e)`

`GEN ell_get_disc(GEN e)`

`GEN ell_get_j(GEN e)`

14.1.3.2 Curves over \mathbf{Q} .

GEN `ellQ_get_N`(GEN `e`) returns the curve conductor

void `ellQ_get_Nfa`(GEN `e`, GEN `*N`, GEN `*faN`) sets `N` to the conductor and `faN` to its factorization

int `ell_is_integral`(GEN `e`) return 1 if `e` is given by an integral model, and 0 otherwise.

long `ellQ_get_CM`(GEN `e`) if `e` has CM by a principal imaginary quadratic order, return its discriminant. Else return 0.

long `ellap_CM_fast`(GEN `e`, ulong `p`, long `CM`) assuming that `p` does not divide the discriminant of `E` (in particular, `E` has good reduction at `p`), and that `CM` is as given by `ellQ_get_CM`, return the trace of Frobenius for E/\mathbf{F}_p . This is meant to quickly compute lots of a_p , esp. when `e` has CM by a principal quadratic order.

long `ellrootno_global`(GEN `e`) returns the global root number $c \in \{-1, 1\}$.

GEN `ellheightoo`(GEN `E`, GEN `P`, long `prec`) given $P = [x, y]$ an affine point on `E`, return

$$\lambda_\infty(P) + \frac{1}{12} \log |\text{disc} E| = \frac{1}{2} \text{real}(z\eta(z)) - \log |\sigma(E, z)| \in \mathbf{R},$$

where $\lambda_\infty(P)$ is the canonical local height at infinity and z is `ellpointtoz`(`E`, `P`). This is computed using Mestre's (quadratically convergent) AGM algorithm.

long `ellorder_Q`(GEN `E`, GEN `P`) return the order of $P \in E(\mathbf{Q})$, using the impossible value 0 for a point of infinite order. Ultimately called by the generic `ellorder` function.

GEN `point_to_a4a6`(GEN `E`, GEN `P`, GEN `p`, GEN `*a4`) given E/\mathbf{Q} , $p \neq 2, 3$ not dividing the discriminant of `E` and $P \in E(\mathbf{Q})$ outside the kernel of reduction, return the image of `P` on the short Weierstrass model $y^2 = x^3 + a_4x + a_6$ isomorphic to the reduction E_p of `E` at `p`. Also set `a4` to the a_4 coefficient in the above model. This function allows quick computations modulo varying primes `p`, avoiding the overhead of `ellinit`(`E`, `p`), followed by a change of coordinates. It produces data suitable for `FpE` routines.

GEN `point_to_a4a6_Fl`(GEN `E`, GEN `P`, ulong `p`, ulong `*pa4`) as `point_to_a4a6`, returning a `FlE`.

GEN `elldatagenerators`(GEN `E`) returns generators for $E(\mathbf{Q})$ extracted from Cremona's table.

GEN `ellanal_globalred`(GEN `e`, GEN `*v`) takes an `ell` over \mathbf{Q} and returns a global minimal model `E` (in `ellinit` form, over \mathbf{Q}) for `e` suitable for analytic computations related to the curve `L` series: it contains `ellglobalred` data, as well as global and local root numbers. If `v` is not `NULL`, set `*v` to the needed change of variable: `NULL` if `e` was already the standard minimal model, such that $E = \text{ellchangecurve}(e, v)$ otherwise. Compared to the direct use of `ellchangecurve` followed by `ellrootno`, this function avoids converting unneeded dynamic data and avoids potential memory leaks (the changed curve would have had to be deleted using `obj_free`). The original curve `e` is updated as well with the same information.

GEN `ellanal_globalred_all`(GEN `e`, GEN `*v`, GEN `*N`, GEN `*tam`) as `ellanal_globalred`; further set `*N` to the curve conductor and `*tam` to the product of the local Tamagawa numbers, including the factor at infinity (multiply by the number of connected components of $e(\mathbf{R})$).

GEN `ellintegralmodel`(GEN `e`, GEN `*pv`) return an integral model for `e` (in `ellinit` form, over \mathbf{Q}). Set `v = NULL` (already integral, we returned `e` itself), else to the variable change $[u, 0, 0, 0]$ making `e` integral. We have $u = 1/t$, $t > 1$.

GEN `ellintegralmodel_i`(GEN `e`, GEN `*pv`) shallow version of `ellintegralmodel`.

Deprecated routines.

GEN `elltors0`(GEN `e`, long `flag`) this function is deprecated; use `elltors`

14.1.3.3 Curves over a number field nf .

Let K be the number field over which E is defined, given by a nf or bnf structure.

GEN `ellnf_get_nf`(GEN `E`) returns the underlying nf .

GEN `ellnf_get_bnf`(GEN `x`) returns NULL if K does not contain a bnf structure, else return the bnf .

GEN `ellnf_vecarea`(GEN `E`) returns the vector of the period lattices areas of all the complex embeddings of E in the same order as `E.nf.roots`.

GEN `ellnf_veceta`(GEN `E`) returns the vector of the quasi-periods of all the complex embeddings of E in the same order as `E.nf.roots`.

GEN `ellnf_vecomega`(GEN `E`) returns the vector of the periods of all the complex embeddings of E in the same order as `E.nf.roots`.

14.1.3.4 Curves over \mathbb{Q}_p .

GEN `ellQp_get_p`(GEN `E`) returns p

long `ellQp_get_prec`(GEN `E`) returns the default p -adic accuracy to which we must compute approximate results attached to E .

GEN `ellQp_get_zero`(GEN `x`) returns $O(p^n)$, where n is the default p -adic accuracy as above.

The following functions are only defined when E has multiplicative reduction (Tate curves):

GEN `ellQp_Tate_uniformization`(GEN `E`, long `prec`) returns a `t_VEC` containing $u^2, u, q, [a, b]$, at p -adic precision `prec`.

GEN `ellQp_u`(GEN `E`, long `prec`) returns u .

GEN `ellQp_u2`(GEN `E`, long `prec`) returns u^2 .

GEN `ellQp_q`(GEN `E`, long `prec`) returns the Tate period q .

GEN `ellQp_ab`(GEN `E`, long `prec`) returns $[a, b]$.

GEN `ellQp_AGM`(GEN `E`, long `prec`) returns $[a, b, R, v]$, where v is an integer, a, b, R are vectors describing the sequence of 2-isogenous curves $E_i : y^2 = x(x + A_i)(x + A_i - B_i)$, $i \geq 1$ converging to the singular curve $E_\infty : y^2 = x^2(x + M)$. We have $a[i] = A[i]p^v$, $b[i] = B[i]p^v$, $R[i] = A_i - B_i$. These are used in `ellpointtoz` and `ellztopoint`.

GEN `ellQp_L`(GEN `E`, long `prec`) returns the \mathcal{L} -invariant L .

GEN `ellQp_root`(GEN `E`, long `prec`) returns e_1 .

14.1.3.5 Curves over a finite field \mathbf{F}_q .

GEN `ellff_get_p`(GEN `E`) returns the characteristic

GEN `ellff_get_field`(GEN `E`) returns p if \mathbf{F}_q is a prime field, and a `t_FFELT` belonging to \mathbf{F}_q otherwise.

GEN `ellff_get_card`(GEN `E`) returns $\#E(\mathbf{F}_q)$

GEN `ellff_get_gens`(GEN `E`) returns a minimal set of generators for $E(\mathbf{F}_q)$.

GEN `ellff_get_group`(GEN `E`) returns `ellgroup`(E).

GEN `ellff_get_m`(GEN `E`) returns the `t_INT` m as needed by the `gen_ellgroup` function (the order of the pairing required to verify a generating set).

GEN `ellff_get_o`(GEN `E`) returns $[d, \text{factor}d]$, where d is the exponent of $E(\mathbf{F}_q)$.

GEN `ellff_get_D`(GEN `E`) returns the elementary divisors for $E(\mathbf{F}_q)$ in a form suitable for `gen_ellgens`: either $[d_1]$ or $[d_1, d_2]$, where d_1 is in `ellff_get_o` format.

$[d, \text{factor}d]$, where d is the exponent of $E(\mathbf{F}_q)$.

GEN `ellff_get_a4a6`(GEN `E`) returns a canonical “short model” for E , and the corresponding change of variable $[u, r, s, t]$. For $p \neq 2, 3$, this is $[A_4, A_6, [u, r, s, t]]$, corresponding to $y^2 = x^3 + A_4x + A_6$, where $A_4 = -27c_4$, $A_6 = -54c_6$, $[u, r, s, t] = [6, 3b_2, 3a_1, 108a_3]$.

- If $p = 3$ and the curve is ordinary ($b_2 \neq 0$), this is $[[b_2], A_6, [1, v, -a_1, -a_3]]$, corresponding to

$$y^2 = x^3 + b_2x^2 + A_6,$$

where $v = b_4/b_2$, $A_6 = b_6 - v(b_4 + v^2)$.

- If $p = 3$ and the curve is supersingular ($b_2 = 0$), this is $[-b_4, b_6, [1, 0, -a_1, -a_3]]$, corresponding to

$$y^2 = x^3 + 2b_4x + b_6.$$

- If $p = 2$ and the curve is ordinary ($a_1 \neq 0$), return $[A_2, A_6, [a_1^{-1}, da_1^{-2}, 0, (a_4 + d^2)a_1^{-1}]]$, corresponding to

$$y^2 + xy = x^3 + A_2x^2 + A_6,$$

where $d = a_3/a_1$, $a_1^2A_2 = (a_2 + d)$ and

$$a_1^6A_6 = d^3 + a_2d^2 + a_4d + a_6 + (a_4^2 + d^4)a_1^{-2}.$$

- If $p = 2$ and the curve is supersingular ($a_1 = 0$, $a_3 \neq 0$), return $[[a_3, A_4, 1/a_3], A_6, [1, a_2, 0, 0]]$, corresponding to

$$y^2 + a_3y = x^3 + A_4x + A_6,$$

where $A_4 = a_2^2 + a_4$, $A_6 = a_2a_4 + a_6$. The value $1/a_3$ is included in the vector since it is frequently needed in computations.

14.1.3.6 Curves over \mathbf{C} . (This includes curves over \mathbf{Q} !)

`long ellR_get_prec(GEN E)` return the default accuracy to which we must compute approximate results attached to E .

`GEN ellR_ab(GEN E, long prec)` return $[a, b]$

`GEN ellR_omega(GEN x, long prec)` return periods $[\omega_1, \omega_2]$.

`GEN ellR_eta(GEN E, long prec)` return quasi-periods $[\eta_1, \eta_2]$.

`GEN ellR_area(GEN x, long prec)` return the area $(\Im(\omega_1\overline{\omega_2}))$.

`GEN ellR_roots(GEN E, long prec)` return $[e_1, e_2, e_3]$. If E is defined over \mathbf{R} , then e_1 is real. If furthermore $\text{disc}E > 0$, then $e_1 > e_2 > e_3$.

`long ellR_get_sign(GEN E)` if E is defined over \mathbf{R} returns the signe of its discriminant, otherwise return 0.

14.1.4 Points.

`int ell_is_inf(GEN z)` tests whether the point z is the point at infinity.

`GEN ellinf()` returns the point at infinity $[0]$.

14.1.5 Change of variables.

`GEN ellchangeinvert(GEN w)` given a change of variables $w = [u, r, s, t]$, returns the inverse change of variables w' , such that if $E' = \text{ellchangecurve}(E, w)$, then $E = \text{ellchangecurve}(E', w')$.

14.1.6 Generic helper functions.

The naming scheme assumes an affine equation $F(x, y) = f(x) - (y^2 + h(x)y) = 0$ in standard Weierstrass form: $f = x^3 + a_2x^2 + a_4x + a_6$, $h = a_1x + a_3$. Unless mentionned otherwise, these routine assume that all arguments are compatible with generic functions of `gadd` or `gmul` type. In particular they do not handle elements in number field in `nfalgtobasis` format.

`GEN ellbasechar(GEN E)` returns the characteristic of the base ring over which E is defined.

`GEN ec_bmodel(GEN E)` returns the polynomial $4x^3 + b_2x^2 + 2b_4x + b_6$.

`GEN ec_f_evalx(GEN E, GEN x)` returns $f(x)$.

`GEN ec_h_evalx(GEN E, GEN x)` returns $h(x)$.

`GEN ec_dFdx_evalQ(GEN E, GEN Q)` returns $3x^2 + 2a_2x + a_4 - a_1y$, where $Q = [x, y]$.

`GEN ec_dFdy_evalQ(GEN E, GEN Q)` returns $-(2y + a_1x + a_3)$, where $Q = [x, y]$.

`GEN ec_dmFdy_evalQ(GEN e, GEN Q)` returns $2y + a_1x + a_3$, where $Q = [x, y]$.

`GEN ec_2divpol_evalx(GEN E, GEN x)` returns $4x^3 + b_2x^2 + 2b_4x + b_6$. This function supports inputs in `nfalgtobasis` format.

`GEN ec_half_deriv_2divpol_evalx(GEN E, GEN x)` returns $6x^2 + b_2x + b_4$.

`GEN ec_3divpol_evalx(GEN E, GEN x)` returns $3x^4 + b_2x^2 + 3b_4x^2 + 3b_6x + b_8$.

14.1.7 Functions to handle elliptic curves over finite fields.

14.1.7.1 Tolerant routines.

GEN `ellap`(GEN `E`, GEN `p`) given a prime number p and an elliptic curve defined over \mathbf{Q} or \mathbf{Q}_p (assumed integral and minimal at p), computes the trace of Frobenius $a_p = p + 1 - \#E(\mathbf{F}_p)$. If E is defined over a non-prime finite field \mathbf{F}_q , ignore p and return $q + 1 - \#E(\mathbf{F}_q)$. When p is implied (E defined over \mathbf{Q}_p or a finite field), p can be omitted (set to `NULL`).

14.1.7.2 Curves defined a non-prime finite field. In this subsection, we assume that `ell_get_type`(E) is `t_ELL_Fq`. (As noted above, a curve defined over $\mathbf{Z}/p\mathbf{Z}$ can be represented as a `t_ELL_Fq`.)

GEN `FF_elltwist`(GEN `E`) returns the coefficients $[a_1, a_2, a_3, a_4, a_6]$ of the quadratic twist of E .

GEN `FF_ellmul`(GEN `E`, GEN `P`, GEN `n`) returns $[n]P$ where n is an integer and P is a point on the curve E .

GEN `FF_ellrandom`(GEN `E`) returns a random point in $E(\mathbf{F}_q)$. This function never returns the point at infinity, unless this is the only point on the curve.

GEN `FF_ellorder`(GEN `E`, GEN `P`, GEN `o`) returns the order of the point P , where o is a multiple of the order of P , or its factorization.

GEN `FF_ellcard`(GEN `E`) returns $\#E(\mathbf{F}_q)$.

GEN `FF_ellcard_SEA`(GEN `E`, long `s`) This function returns $\#E(\mathbf{F}_q)$, using the Schoof-Elkies-Atkin algorithm. Assume $p \neq 2, 3$. The parameter s has the same meaning as in `Fp_ellcard_SEA`.

GEN `FF_ellgens`(GEN `E`) returns the generators of the group $E(\mathbf{F}_q)$.

GEN `FF_elllog`(GEN `E`, GEN `P`, GEN `G`, GEN `o`) Let G be a point of order o , return e such that $[e]P = G$. If e does not exists, the result is undefined.

GEN `FF_ellgroup`(GEN `E`, GEN `*pm`) returns the structure of the Abelian group $E(\mathbf{F}_q)$ and set `*pm` to m (see `gen_ellgens`).

GEN `FF_ellweilpairing`(GEN `E`, GEN `P`, GEN `Q`, GEN `m`) returns the Weil pairing of the points of m -torsion P and Q .

GEN `FF_elltatepairing`(GEN `E`, GEN `P`, GEN `Q`, GEN `m`) returns the Tate pairing of P and Q , where $[m]P = 0$.

14.2 Arithmetic on elliptic curve over a finite field in simple form.

The functions in this section no longer operate on elliptic curve structures, as seen up to now. They are used to implement those higher-level functions without using cached information and thus require suitable explicitly enumerated data.

14.2.1 Helper functions.

GEN `elltrace_extension`(GEN `t`, long `n`, GEN `q`) Let E some elliptic curve over \mathbf{F}_q such that the trace of the Frobenius is t , returns the trace of the Frobenius over \mathbf{F}_q^n .

14.2.2 Elliptic curves over \mathbf{F}_p , $p > 3$.

Let p a prime number and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, with a_4 and a_6 in \mathbf{F}_p . A \mathbf{FpE} is a point of $E(\mathbf{F}_p)$. Since an affine point and a_4 determine an unique a_6 , most functions do not take a_6 as an argument. A \mathbf{FpE} is either the point at infinity (`ellinf()`) or a \mathbf{FpV} which two components. The parameters a_4 and a_6 are given as `t_INTs` when required.

`GEN Fp_ellj(GEN a4, GEN a6, GEN p)` returns the j -invariant of the curve E .

`int Fp_elljissupersingular(GEN j, GEN p)` returns 1 if j is the j -invariant of a supersingular curve over \mathbf{F}_p , 0 otherwise.

`GEN Fp_ellcard(GEN a4, GEN a6, GEN p)` returns the cardinality of the group $E(\mathbf{F}_p)$.

`GEN Fp_ellcard_SEA(GEN a4, GEN a6, GEN p, long s)` This function returns $\#E(\mathbf{F}_p)$, using the Schoof-Elkies-Atkin algorithm. If the `seadata` package is installed, the function will be faster.

The extra flag `s`, if set to a non-zero value, causes the computation to return `gen_0` (an impossible cardinality) if one of the small primes ℓ divides the curve order but does not divide s . For cryptographic applications, where one is usually interested in curves of prime order, setting $s = 1$ efficiently weeds out most uninteresting curves; if curves of order a power of 2 times a prime are acceptable, set $s = 2$. If moreover `s` is negative, similar checks are performed for the twist of the curve.

`GEN Fp_ffellcard(GEN a4, GEN a6, GEN q, long n, GEN p)` returns the cardinality of the group $E(\mathbf{F}_q)$ where $q = p^n$.

`GEN Fp_ellgroup(GEN a4, GEN a6, GEN N, GEN p, GEN *pm)` returns the group structure D of the group $E(\mathbf{F}_p)$, which is assumed to be of order N and set `*pm` to m .

`GEN Fp_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN p)` returns generators of the group $E(\mathbf{F}_p)$ with the base change `ch` (see `FpE.changepoint`), where D and m are as returned by `Fp_ellgroup`.

`GEN Fp_elldivpol(GEN a4, GEN a6, long n, GEN p)` returns the n -division polynomial of the elliptic curve E .

`void Fp_elltwist(GEN a4, GEN a6, GEN p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E .

14.2.3 \mathbf{FpE} .

`GEN FpE_add(GEN P, GEN Q, GEN a4, GEN p)` returns the sum $P + Q$ in the group $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$, for any value of a_6 compatible with the points given.

`GEN FpE_sub(GEN P, GEN Q, GEN a4, GEN p)` returns $P - Q$.

`GEN FpE_dbl(GEN P, GEN a4, GEN p)` returns $2.P$.

`GEN FpE_neg(GEN P, GEN p)` returns $-P$.

`GEN FpE_mul(GEN P, GEN n, GEN a4, GEN p)` return $n.P$.

`GEN FpE_changepoint(GEN P, GEN m, GEN a4, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a \mathbf{FpV}).

`GEN FpE_changepointinv(GEN P, GEN m, GEN a4, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a \mathbf{FpV}).

GEN random_FpE(GEN a4, GEN a6, GEN p) returns a random point on $E(\mathbf{F}_p)$, where E is defined by $E : y^2 = x^3 + a_4x + a_6$.

GEN FpE_order(GEN P, GEN o, GEN a4, GEN p) returns the order of P in the group $E(\mathbf{F}_p)$, where o is a multiple of the order of P , or its factorization.

GEN FpE_log(GEN P, GEN G, GEN o, GEN a4, GEN p) Let G be a point of order o , return e such that $e.P = G$. If e does not exist, the result is currently undefined.

GEN FpE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Tate pairing of the point of m -torsion P and the point Q .

GEN FpE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Weil pairing of the points of m -torsion P and Q .

GEN FpE_to_mod(GEN P, GEN p) returns P as a vector of $\mathfrak{t_INTMODs}$.

GEN RgE_to_FpE(GEN P, GEN p) returns the FpE obtained by applying Rg_to_Fp coefficientwise.

14.2.4 Fle. Let p be a prime ulong, and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are ulong. A Fle is either the point at infinity (`ellinf()`), or a Flv with two components $[x, y]$.

long Fl_elltrace(ulong a4, ulong a6, ulong p) returns the trace t of the Frobenius of $E(\mathbf{F}_p)$. The cardinality of $E(\mathbf{F}_p)$ is thus $p + 1 - t$, which might not fit in an ulong.

long Fl_elltrace_CM(long CM, ulong a4, ulong a6, ulong p) as Fl_elltrace. If CM is 0, use the standard algorithm; otherwise assume the curve has CM by a principal imaginary quadratic order of discriminant CM and use a faster algorithm. Useful when the curve is the reduction of E/\mathbf{Q} , which has CM by a principal order, and we need the trace of Frobenius for many distinct p , see `ellQ_get_CM`.

ulong Fl_elldisc(ulong a4, ulong a6, ulong p) returns the discriminant of the curve E .

ulong Fl_elldisc_pre(ulong a4, ulong a6, ulong p, ulong pi) returns the discriminant of the curve E , assuming pi is the pseudo inverse of p .

ulong Fl_ellj(ulong a4, ulong a6, ulong p) returns the j -invariant of the curve E .

ulong Fl_ellj_pre(ulong a4, ulong a6, ulong p, ulong pi) returns the j -invariant of the curve E , assuming pi is the pseudo inverse of p .

void Fl_ellj_to_a4a6(ulong j, ulong p, ulong *pa4, ulong *pa6) sets *pa4 to a_4 and *pa6 to a_6 where a_4 and a_6 define a fixed elliptic curve with j -invariant j .

void Fl_elltwist(ulong a4, ulong a6, ulong p, ulong *pA4, ulong *pA6) set *pA4 to A_4 and *pA6 to A_6 where A_4 and A_6 define the twist of E .

void Fl_elltwist_disc(ulong a4, ulong a6, ulong D, ulong p, ulong *pA4, ulong *pA6) sets *pA4 to A_4 and *pA6 to A_6 where A_4 and A_6 define the twist of E by the discriminant D .

GEN Fle_add(GEN P, GEN Q, ulong a4, ulong p)

GEN Fle_dbl(GEN P, ulong a4, ulong p)

GEN Fle_sub(GEN P, GEN Q, ulong a4, ulong p)

GEN Fle_mul(GEN P, GEN n, ulong a4, ulong p)

```

GEN Fle_mulu(GEN P, ulong n, ulong a4, ulong p)
GEN Fle_order(GEN P, GEN o, ulong a4, ulong p)
GEN Fle_log(GEN P, GEN G, GEN o, ulong a4, ulong p)
GEN random_Fle(ulong a4, ulong a6, ulong p)
GEN random_Fle_pre(ulong a4, ulong a6, ulong p, ulong pi)
GEN Fle_changepoint(GEN x, GEN ch, ulong p), ch is assumed to give the change of coordinates
[u, r, s, t] as a t_VECSMALL.
GEN Fle_changepointinv(GEN x, GEN ch, ulong p), as Fle_changepoint

```

14.2.5 FpJ.

Let p be a prime t_INT , and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are t_INT . A FpJ is a FpV with three components $[x, y, z]$, representing the affine point $[x/z^2, y/z^3]$ in Jacobian coordinates, the point at infinity being represented by $[1, 1, 0]$. The following must hold: $y^2 = x^3 + a_4xz^4 + a_6z^6$. For all non-zero u , the points $[u^2x, u^3y, uz]$ and $[x, y, z]$ are representing the same affine point.

```

GEN FpJ_add(GEN P, GEN Q, GEN a4, GEN p)
GEN FpJ_dbl(GEN P, GEN a4, GEN p)
GEN FpJ_mul(GEN P, GEN n, GEN a4, GEN p);
GEN FpJ_neg(GEN P, GEN p) return  $-P$ .
GEN FpJ_to_FpE(GEN P, GEN p) return the corresponding FpE.
GEN FpE_to_FpJ(GEN P) return the corresponding FpJ.

```

14.2.6 Flj.

Below, pi is assumed to be the precomputed inverse of p .

```

GEN Fle_to_Flj(GEN P) convert a Fle to an equivalent Flj.
GEN Flj_to_Fle_pre(GEN P) convert a Flj to the equivalent Fle.
GEN Flj_add_pre(GEN P, GEN Q, ulong a4, ulong p, ulong pi)
GEN Flj_dbl_pre(GEN P, ulong a4, ulong p, ulong pi)
GEN Flj_neg(GEN P, ulong p) return  $-P$ .
GEN Flj_mulu_pre(GEN P, ulong n, ulong a4, ulong p, ulong pi)
GEN random_Flj_pre(ulong a4, ulong a6, ulong p, ulong pi)

```

14.2.7 Elliptic curves over \mathbf{F}_{2^n} . Let T be an irreducible $\mathbf{F}_2[x]$ and E the elliptic curve given by either the equation $E : y^2 + x * y = x^3 + a_2x^2 + a_6$, where a_2, a_6 are $\mathbf{F}_2[x]$ in $\mathbf{F}_2[X]/(T)$ (ordinary case) or $E : y^2 + a_3 * y = x^3 + a_4x + a_6$, where a_3, a_4, a_6 are $\mathbf{F}_2[x]$ in $\mathbf{F}_2[X]/(T)$ (supersingular case).

A $\mathbf{F}_2[x]E$ is a point of $E(\mathbf{F}_2[X]/(T))$. In the supersingular case, the parameter a_2 is actually the $\mathbf{t_VEC} [a_3, a_4, a_3^{-1}]$.

`GEN F2xq_ellcard(GEN a2, GEN a6, GEN T)` Return the order of the group $E(\mathbf{F}_2[X]/(T))$.

`GEN F2xq_ellgroup(GEN a2, GEN a6, GEN N, GEN T, GEN *pm)` Return the group structure D of the group $E(\mathbf{F}_2[X]/(T))$, which is assumed to be of order N and set $*pm$ to m .

`GEN F2xq_ellgens(GEN a2, GEN a6, GEN ch, GEN D, GEN m, GEN T)` Returns generators of the group $E(\mathbf{F}_2[X]/(T))$ with the base change ch (see `F2xqE.changept`), where D and m are as returned by `F2xq_ellgroup`.

`void F2xq_elltwist(GEN a4, GEN a6, GEN T, GEN *a4t, GEN *a6t)` sets $*a4t$ and $*a6t$ to the parameters of the quadratic twist of E .

14.2.8 $\mathbf{F}_2[x]E$.

`GEN F2xqE_changept(GEN P, GEN m, GEN a2, GEN T)` returns the image Q of the point P on the curve $E : y^2 + x * y = x^3 + a_2x^2 + a_6$ by the coordinate change m (which is a $\mathbf{F}_2[x]V$).

`GEN F2xqE_changeptinv(GEN P, GEN m, GEN a2, GEN T)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a $\mathbf{F}_2[x]V$).

`GEN F2xqE_add(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_sub(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_dbl(GEN P, GEN a2, GEN T)`

`GEN F2xqE_neg(GEN P, GEN a2, GEN T)`

`GEN F2xqE_mul(GEN P, GEN n, GEN a2, GEN T)`

`GEN random_F2xqE(GEN a2, GEN a6, GEN T)`

`GEN F2xqE_order(GEN P, GEN o, GEN a2, GEN T)` returns the order of P in the group $E(\mathbf{F}_2[X]/(T))$, where o is a multiple of the order of P , or its factorization.

`GEN F2xqE_log(GEN P, GEN G, GEN o, GEN a2, GEN T)` Let G be a point of order o , return e such that $e.P = G$. If e does not exist, the result is currently undefined.

`GEN F2xqE_tatepairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Tate pairing of the point of m -torsion P and the point Q .

`GEN F2xqE_weilpairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Weil pairing of the points of m -torsion P and Q .

`GEN RgE_to_F2xqE(GEN P, GEN T)` returns the $\mathbf{F}_2[x]E$ obtained by applying `Rg_to_F2xq` coefficientwise.

14.2.9 Elliptic curves over \mathbf{F}_q , small characteristic $p > 2$. Let $p > 2$ be a prime `ulong`, T an irreducible `Flx` mod p , and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$, where a_4 and a_6 are `Flx` in $\mathbf{F}_p[X]/(T)$. A `FlxqE` is a point of $E(\mathbf{F}_p[X]/(T))$.

In the special case $p = 3$, ordinary elliptic curves ($j(E) \neq 0$) cannot be represented as above, but admit a model $E : y^2 = x^3 + a_2x^2 + a_6$ with a_2 and a_6 being `Flx` in $\mathbf{F}_3[X]/(T)$. In that case, the parameter `a2` is actually stored as a `t_VEC`, $[a_2]$, to avoid ambiguities.

`GEN Flxq_ellj(GEN a4, GEN a6, GEN T, ulong p)` returns the j -invariant of the curve E .

`void Flxq_ellj_to_a4a6(GEN j, GEN T, ulong p, GEN *pa4, GEN *pa6)` sets `*pa4` to a_4 and `*pa6` to a_6 where a_4 and a_6 define a fixed elliptic curve with j -invariant j .

`GEN Flxq_ellcard(GEN a4, GEN a6, GEN T, ulong p)` returns the order of $E(\mathbf{F}_p[X]/(T))$.

`GEN Flxq_ellgroup(GEN a4, GEN a6, GEN N, GEN T, ulong p, GEN *pm)` returns the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and sets `*pm` to m .

`GEN Flxq_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, ulong p)` returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FlxqE_changepoint`), where D and m are as returned by `Flxq_ellgroup`.

`void Flxq_elltwist(GEN a4, GEN a6, GEN T, ulong p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E .

14.2.10 `FlxqE`.

Let $p > 2$ be a prime number.

`GEN FlxqE_changepoint(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_changepointinv(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a `FlxqV`).

`GEN FlxqE_add(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_sub(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_dbl(GEN P, GEN a4, GEN T, ulong p)`

`GEN FlxqE_neg(GEN P, GEN T, ulong p)`

`GEN FlxqE_mul(GEN P, GEN n, GEN a4, GEN T, ulong p)`

`GEN random_FlxqE(GEN a4, GEN a6, GEN T, ulong p)`

`GEN FlxqE_order(GEN P, GEN o, GEN a4, GEN T, ulong p)` returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P , or its factorization.

`GEN FlxqE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, ulong p)` Let G be a point of order o , return e such that $e.P = G$. If e does not exist, the result is currently undefined.

`GEN FlxqE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Tate pairing of the point of m -torsion P and the point Q .

`GEN FlxqE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Weil pairing of the points of m -torsion P and Q .

`GEN RgE_to_FlxqE(GEN P, GEN T, ulong p)` returns the `FlxqE` obtained by applying `RgE_to_FlxqE` coefficientwise.

14.2.11 Elliptic curves over \mathbf{F}_q , large characteristic .

Let $p > 3$ be a prime number, T an irreducible polynomial mod p , and E the elliptic curve given by the equation $E : y^2 = x^3 + a_4x + a_6$ with a_4 and a_6 in $\mathbf{F}_p[X]/(T)$. A $\mathbf{F}_p\text{XQE}$ is a point of $E(\mathbf{F}_p[X]/(T))$.

`GEN FpXQ_ellj(GEN a4, GEN a6, GEN T, GEN p)` returns the j -invariant of the curve E .

`int FpXQ_elljissupersingular(GEN j, GEN T, GEN p)` returns 1 if j is the j -invariant of a supersingular curve over $\mathbf{F}_p[X]/(T)$, 0 otherwise.

`GEN FpXQ_ellcard(GEN a4, GEN a6, GEN T, GEN p)` returns the order of $E(\mathbf{F}_p[X]/(T))$.

`GEN Fq_ellcard_SEA(GEN a4, GEN a6, GEN q, GEN T, GEN p, long s)` This function returns $\#E(\mathbf{F}_p[X]/(T))$, using the Schoof-Elkies-Atkin algorithm. Assume $p \neq 2, 3$, and q is the cardinality of $\mathbf{F}_p[X]/(T)$. The parameter s has the same meaning as in `Fp_ellcard_SEA`. If the `seadata` package is installed, the function will be faster.

`GEN FpXQ_ellgroup(GEN a4, GEN a6, GEN N, GEN T, GEN p, GEN *pm)` Return the group structure D of the group $E(\mathbf{F}_p[X]/(T))$, which is assumed to be of order N and set `*pm` to m .

`GEN FpXQ_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, GEN p)` Returns generators of the group $E(\mathbf{F}_p[X]/(T))$ with the base change `ch` (see `FpXQE_changepoint`), where D and m are as returned by `FpXQ_ellgroup`.

`GEN FpXQ_elldivpol(GEN a4, GEN a6, long n, GEN T, GEN p)` returns the n -division polynomial of the elliptic curve E .

`GEN Fq_elldivpolmod(GEN a4, GEN a6, long n, GEN h, GEN T, GEN p)` returns the n -division polynomial of the elliptic curve E modulo the polynomial h .

`void FpXQ_elltwist(GEN a4, GEN a6, GEN T, GEN p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of E .

14.2.12 $\mathbf{F}_p\text{XQE}$.

`GEN FpXQE_changepoint(GEN P, GEN m, GEN a4, GEN T, GEN p)` returns the image Q of the point P on the curve $E : y^2 = x^3 + a_4x + a_6$ by the coordinate change m (which is a $\mathbf{F}_p\text{XQV}$).

`GEN FpXQE_changepointinv(GEN P, GEN m, GEN a4, GEN T, GEN p)` returns the image Q on the curve $E : y^2 = x^3 + a_4x + a_6$ of the point P by the inverse of the coordinate change m (which is a $\mathbf{F}_p\text{XQV}$).

`GEN FpXQE_add(GEN P, GEN Q, GEN a4, GEN T, GEN p)`

`GEN FpXQE_sub(GEN P, GEN Q, GEN a4, GEN T, GEN p)`

`GEN FpXQE_dbl(GEN P, GEN a4, GEN T, GEN p)`

`GEN FpXQE_neg(GEN P, GEN T, GEN p)`

`GEN FpXQE_mul(GEN P, GEN n, GEN a4, GEN T, GEN p)`

`GEN random_FpXQE(GEN a4, GEN a6, GEN T, GEN p)`

`GEN FpXQE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, GEN p)` Let G be a point of order o , return e such that $e.P = G$. If e does not exist, the result is currently undefined.

`GEN FpXQE_order(GEN P, GEN o, GEN a4, GEN T, GEN p)` returns the order of P in the group $E(\mathbf{F}_p[X]/(T))$, where o is a multiple of the order of P , or its factorization.

GEN `FpXQE_tatepairing`(GEN `P`, GEN `Q`, GEN `m`, GEN `a4`, GEN `T`, GEN `p`) returns the Tate pairing of the point of m -torsion P and the point Q .

GEN `FpXQE_weilpairing`(GEN `P`, GEN `Q`, GEN `m`, GEN `a4`, GEN `T`, GEN `p`) returns the Weil pairing of the points of m -torsion P and Q .

GEN `RgE_to_FpXQE`(GEN `P`, GEN `T`, GEN `p`) returns the `FpXQE` obtained by applying `Rg_to_FpXQ` coefficientwise.

14.3 Functions related to modular polynomials.

Variants of `polmodular`, returning the modular polynomial of prime level L for the invariant coded by `inv` (0: j , 1: Weber- f , see `polclass` for the full list).

GEN `polmodular_ZXX`(long `L`, long `inv`, long `xvar`, long `yvar`) returns a bivariate polynomial in variables `xvar` and `yvar`.

GEN `polmodular_ZM`(long `L`, long `inv`) returns a matrix of (integral) coefficients.

GEN `Fp_polmodular_evalx`(long `L`, long `inv`, GEN `J`, GEN `p`, long `v`, int `derivs`) returns the modular polynomial evaluated at J modulo the prime p in the variable v (if `derivs` is non-zero, returns a vector containing the modular polynomial and its first and second derivatives, all evaluated at J modulo p).

14.3.1 Functions related to modular invariants.

void `check_modinv`(long `inv`) report an error if `inv` is not a valid code for a modular invariant.

int `modinv_good_disc`(long `inv`, long `D`) test whether the invariant `inv` is defined for the discriminant `D`.

int `modinv_good_prime`(long `inv`, long `D`) test whether the invariant `inv` is defined for the prime `p`.

long `modinv_height_factor`(long `inv`) return the height factor of the modular invariant `inv` with respect to the j -invariant. This is an integer n such that the j -invariant is asymptotically of the order of the n -th power of the invariant `inv`.

long `modinv_is_Weber`(long `inv`) test whether the invariant `inv` is a power of Weber f .

long `modinv_is_double_eta`(long `inv`) test whether the invariant `inv` is a double η quotient.

long `disc_best_modinv`(long `D`) the integer D being a negative discriminant, return the modular invariant compatible with D with the highest height factor.

GEN `Fp_modinv_to_j`(GEN `x`, long `inv`, GEN `p`) Let Φ the modular equation between j and the modular invariant `inv`, return y such that $\Phi(y, x) = 0 \pmod{p}$.

14.4 Other curves.

The following functions deal with hyperelliptic curves in weighted projective space $\mathbf{P}_{(1,d,1)}$, with coordinates (x, y, z) and a model of the form $y^2 = T(x, z)$, where T is homogeneous of degree $2d$, and squarefree. Thus the curve is nonsingular of genus $d - 1$.

`long hyperell_locally_soluble(GEN T, GEN p)` assumes that $T \in \mathbf{Z}[X]$ is integral. Returns 1 if the curve is locally soluble over \mathbf{Q}_p , 0 otherwise.

`long nf_hyperell_locally_soluble(GEN nf, GEN T, GEN pr)` let K be a number field, attached to `nf`, `pr` a *prid* attached to some maximal ideal \mathfrak{p} ; assumes that $T \in \mathbf{Z}_K[X]$ is integral. Returns 1 if the curve is locally soluble over $K_{\mathfrak{p}}$.

Chapter 15: *L*-functions

15.1 Accessors.

```
long is_linit(GEN data)
GEN ldata_get_an(GEN ldata)
GEN ldata_get_dual(GEN ldata)
long ldata_isreal(GEN ldata)
GEN ldata_get_gammavec(GEN ldata)
long ldata_get_degree(GEN ldata)
long ldata_get_k(GEN ldata)
GEN ldata_get_conductor(GEN ldata)
GEN ldata_get_rootno(GEN ldata)
GEN ldata_get_residue(GEN ldata)
GEN ldata_vecan(GEN ldata, long L, long prec)
long ldata_get_type(GEN ldata)
long linit_get_type(GEN linit)
GEN linit_get_ldata(GEN linit)
GEN linit_get_tech(GEN linit)
GEN lfun_get_domain(GEN tech)
GEN lfun_get_dom(GEN tech)
long lfun_get_bitprec(GEN tech)
GEN lfun_get_factgammavec(GEN tech)
GEN lfun_get_step(GEN tech)
GEN lfun_get_pol(GEN tech)
GEN lfun_get_Residue(GEN tech)
GEN lfun_get_k2(GEN tech)
GEN lfun_get_w2(GEN tech)
GEN lfun_get_expot(GEN tech)
long lfun_get_bitprec(GEN tech)
```

GEN lfunprod_get_fact(GEN tech)
 GEN theta_get_an(GEN tdata)
 GEN theta_get_K(GEN tdata)
 GEN theta_get_R(GEN tdata)
 long theta_get_bitprec(GEN tdata)
 long theta_get_m(GEN tdata)
 GEN theta_get_tdom(GEN tdata)
 GEN theta_get_sqrtN(GEN tdata)

15.2 Conversions and constructors.

GEN lfunmisc_to_ldata(GEN ldata)
 GEN lfunmisc_to_ldata_shallow(GEN ldata)
 GEN lfunrttopoles(GEN r)
 int sdomain_isincl(GEN dom, GEN dom0)

15.3 Variants of GP functions.

GEN lfun(GEN ldata, GEN s, long bitprec)
 GEN lfuninit(GEN ldata, GEN dom, long der, long bitprec)
 GEN lfuninit_make(long t, GEN ldata, GEN molin, GEN domain)
 GEN lfunlambda(GEN ldata, GEN s, long bitprec)
 long lfunthetacost(GEN ldata, GEN tdom, long m, long bitprec): lfunthetacost0 when
 the first argument is known to be an Ldata.
 GEN lfunthetacheckinit(GEN data, GEN tinf, long m, long bitprec)
 GEN lfunrootno(GEN data, long bitprec)
 GEN lfunzetakinit(GEN pol, GEN dom, long der, long flag, long bitprec)
 GEN lfunellmfpeters(GEN E, long bitprec)
 GEN ellanalyticrank(GEN E, long prec) DEPRECATED.
 GEN ellL1(GEN E, long prec) DEPRECATED.

15.4 Inverse Mellin transforms of Gamma products.

GEN gammamellininv(GEN Vga, GEN s, long m, long bitprec)

GEN gammamellininvinit(GEN Vga, long m, long bitprec)

GEN gammamellininvrt(GEN K, GEN s, long bitprec)

double dbllambertW0(double a)

double dbllambertW_1(double a)

double dblllemma526(double a, double b, double c, long B)

double dbllcoro526(double a, double c, long B)

Chapter 16: Modular symbols

`void checkms(GEN W)` raise an exception if W is not an *ms* structure from `msinit`.

`void checkmspadic(GEN W)` raise an exception if W is not an *mspadic* structure from `mspadicinit`.

Variants of `mfnucusp` :

`ulong mfnucusp(ulong n)`

`GEN mfnucusp_fact(GEN fa)` where `fa` is `factor(n)`.

`ulong mfnucusp_fact(ulong n)` where `fa` is `factoru(n)`.

Chapter 17: Modular forms

17.1 Implementation of public data structures.

`void checkMF(GEN mf)` raise an exception if the argument is not a modular form space.

`GEN checkMF_i(GEN mf)` return the underlying modular form space if `mf` is either directly a modular form space from `mfini` or a symbol from `mfsymbol`. Return `NULL` otherwise.

`int checkmf_i(GEN mf)` return 1 if the argument is a modular form and 0 otherwise.

17.1.1 Accessors for modular form spaces.

Shallow functions; assume that their argument is a modular form space is created by `mfini` and checked using `checkMF`.

`GEN MF_get_gN(GEN mf)` return the level N as a `t_INT`.

`long MF_get_N(GEN mf)` return the level N as a `long`.

`GEN MF_get_gk(GEN mf)` return the level k as a `t_INT`.

`long MF_get_k(GEN mf)` return the level k as a `long`.

`long MF_get_r(GEN mf)` assuming the level is a half-integer, return the integer $r = k - (1/2)$.

`GEN MF_get_CHI(GEN mf)` return the nebentypus χ , which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either ± 1 or `t_POLMOD` in t .

`long MF_get_space(GEN mf)` returns the space type, corresponding to `mfini`'s space flag. The current list is

`mf_NEW, mf_CUSP, mf_OLD, mf_EISEN, mf_FULL`

GEN MF_get_basis(GEN mf) return the \mathbf{Q} -basis of the space, concatenation of MF_get_E and MF_get_S, in this order; the forms have coefficients in $\mathbf{Q}(\chi)$. Low-level version of mfbasis.

long MF_get_dim(GEN mf) returns the dimension d of the space. It is the cardinality of MF_get_basis.

GEN MF_get_E(GEN mf) returns a \mathbf{Q} -basis for the subspace spanned by Eisenstein series in the space; the forms have coefficients in $\mathbf{Q}(\chi)$.

GEN MF_get_S(GEN mf) returns a \mathbf{Q} -basis for the cuspidal subspace in the space; the forms have coefficients in $\mathbf{Q}(\chi)$.

GEN MF_get_fields(GEN mf) returns the vector of polynomials defining each Galois orbit of newforms over $\mathbf{Q}(\chi)$. Uses memoization: a first call splits the space and may be costly; subsequent calls return the cached result.

GEN MF_get_newforms(GEN mf) returns a vector vF containing the coordinates of the eigenforms on MF_get_basis (mftobasis form). Low-level version of mfeigenbasis, whose elements are recovered as mflinear(mf, gel(vF, i)). Uses memoization, sharing the same data as MF_get_fields. Note that it is much more efficient to use mfcoefs(mf,) then multiply by this vector than to compute the coefficients of eigenforms from mfeigenbasis individually.

The following accessors are technical,

GEN MF_get_M(GEN mf) the $(1 + m) \times d$ matrix whose j -th column contain the coefficients of the j -th entry in MF_get_basis, m is the optimal “Sturm bound” for the space: the maximum of the $v_\infty(f)$ over non-zero forms. It has entries in $\mathbf{Q}(\chi)$.

GEN MF_get_Mindex(GEN mf) is a t_VECSMALL containing d row indices, the corresponding rows of M form an invertible matrix M_0 .

GEN MF_get_Minv(GEN mf) the inverse of M_0 in a form suitable for fast multiplication.

GEN MFcusp_get_vMjd(GEN mf) valid only for a full *cuspidal* space. Then the functions in MF_get_S are of the form $B_d T_j Tr_M^{new}$. This returns the vector of triples (t_VECSMALL) $[M, j, d]$, in the same order.

GEN MFnew_get_vj(GEN mf) valid only for a *new* space. Then the functions in MF_get_S are of the form $T_j Tr_N^{new}$. This returns a t_VECSMALL of the Hecke indices j , in the same order.

17.1.2 Accessors for individual modular forms.

GEN mf_get_gN(GEN F) return the level of F , which may be a multiple of the conductor, as a t_INT

long mf_get_n(GEN F) return the level as a long.

GEN mf_get_gk(GEN F) return the weight of F as a t_INT or a t_FRAC with denominator 2 (half-integral weight).

long mf_get_k(GEN F) return the weight as a long; if the weight is not integral, this raises an exception.

long mf_get_r(GEN F) assuming F is a modular form of half-integral weight $k = (2r + 1)/2$, return $r = k - (1/2)$.

GEN mf_get_CHI(GEN F) return the nebentypus, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either ± 1 or t_POLMOD in t .

GEN `mf_get_field`(GEN F) return the polynomial (in variable y) defining $\mathbf{Q}(f)$ over $\mathbf{Q}(\chi)$.

GEN `mf_get_NK`(GEN F) return the tag attached to F : a vector containing `gN`, `gk`, `CHI`, `field`. Never use its component directly, use individual accessors as above.

long `mf_get_type`(GEN F) returns a symbolic name for the constructor used to create the form, e.g. `t_MF_EISEN` for a general Eisenstein series. A form has a recursive structure represented by a tree: its definition may involve other forms, e.g. the tree attached to $T_n f$ contains f as a subtree. Such trees have *leaves*, forms which do not contain a strict subtree, e.g. `t_MF_DELTA` is a leaf, attached to Ramanujan's Δ .

Here is the current list of types; since the names are liable to change, they are not documented at this point. Use `mfdescribe` to visualize their mathematical structure.

```
/*leaves*/
  t_MF_CONST, t_MF_EISEN, t_MF_Ek, t_MF_DELTA, t_MF_ETAQUO, t_MF_ELL,
  t_MF_DIHEDRAL, t_MF_THETA, t_MF_TRACE, t_MF_NEWTRACE,
/*recursive*/
  t_MF_MUL, t_MF_POW, t_MF_DIV, t_MF_BRACKET, t_MF_LINEAR, t_MF_LINEAR_BHN,
  t_MF_SHIFT, t_MF_DERIV, t_MF_DERIVE2, t_MF_TWIST, t_MF_HECKE,
  t_MF_BD,
```

17.1.3 Nebentypus. The characters stored in modular forms and modular form spaces have a special structure. One can recover the parameters of an ordinary Dirichlet character by `G = gel(CHI,1)` (the underlying `znstar`) and `chi = gel(CHI,2)` (the underlying character in `znconreylog` form).

long `mfcharmodulus`(GEN CHI) the modulus of χ .

long `mfcharorder`(GEN CHI) the order of χ .

GEN `mfcharpol`(GEN CHI) the cyclotomic polynomial Φ_n defining $\mathbf{Q}(\chi)$, always normalized so that n is not 2 mod 4.

17.1.4 Miscellaneous functions.

long `mfnewdim`(long N, long k, GEN CHI) dimension of the new part of the cuspidal space.

long `mfcuspdim`(long N, long k, GEN CHI) dimension of the cuspidal space.

long `mfoldddim`(long N, long k, GEN CHI) dimension of the old part of the cuspidal space.

long `mfeisensteindim`(long N, long k, GEN CHI) dimension of the Eisenstein subspace.

long `mffulldim`(long N, long k, GEN CHI) dimension of the full space.

GEN `mfeisensteinspaceinit`(GEN NK)

GEN `mfdiv_val`(GEN F, GEN G, long vG)

GEN `mfembed`(GEN E, GEN v)

GEN `mfmatembed`(GEN E, GEN v)

GEN `mfvecembed`(GEN E, GEN v)

long `mfsturmNgk`(long N, GEN k)

long `mfsturmNk`(long N, long k)

long mfsturm_mf(GEN mf)
long mfiscuspidal(GEN mf, GEN F)
GEN mftobasisES(GEN mf, GEN F)
GEN mftocol(GEN F, long lim, long d)
GEN mfvectomat(GEN vF, long lim, long d)

Chapter 18: Plots

A `PARI_plot` canvas is a record of dimensions, with the following fields:

```
long width; /* window width */
long height; /* window height */
long hunit; /* length of horizontal 'ticks' */
long vunit; /* length of vertical 'ticks' */
long fwidth; /* font width */
long fheight; /* font height */
void (*draw)(PARI_plot *T, GEN w, GEN x, GEN y);
```

The `draw` method performs the actual drawing of a `t_VECSMALL` `w` (rectwindow indices); x and y are `t_VECSMALL`s of the same length and rectwindow $w[i]$ is drawn with its upper left corner at offset $(x[i], y[i])$. No plot engine is available in `libpari` by default, since this would introduce a dependency on extra graphical libraries. See the files `src/graph/plot*` for basic implementations of various plot engines: `plotsvg` is particularly simple (`draw` is a 1-liner).

`void pari_set_plot_engine(void (*T)(PARI_plot *))` installs the graphical engine T and initializes the graphical subsystem. No routine in this chapter will work without this initialization.

`void pari_kill_plot_engine(void)` closes the graphical subsystem and frees the resources it occupies.

18.0.5 Highlevel function. Those functions plot $f(E, x)$ for $x \in [a, b]$, using n regularly spaced points (by default).

`GEN ploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)` draw physically.

`GEN plotrecth(void *E, GEN(*f)(void*, GEN), long w, GEN a, GEN b, ulong flags, long n, long prec)` draw in rectwindow w .

18.0.6 Function .

`void plotbox(long ne, GEN gx2, GEN gy2)`

`void plotclip(long rect)`

`void plotcolor(long ne, long color)`

`void plotcopy(long source, long dest, GEN xoff, GEN yoff, long flag)`

`GEN plotcursor(long ne)`

`void plotdraw(GEN list, long flag)`

`GEN plothraw(GEN listx, GEN listy, long flag)`

`GEN plotsizes(long flag)`

`void plotinit(long ne, GEN x, GEN y, long flag)`

`void plotkill(long ne)`

```

void plotline(long ne, GEN x2, GEN y2)
void plotlines(long ne, GEN listx, GEN listy, long flag)
void plotlinetype(long ne, long t)
void plotmove(long ne, GEN x, GEN y)
void plotpoints(long ne, GEN listx, GEN listy)
void plotpointsize(long ne, GEN size)
void plotpointtype(long ne, long t)
void plotrbox(long ne, GEN x2, GEN y2)
GEN plotrecthraw(long ne, GEN data, long flags)
void plotrline(long ne, GEN x2, GEN y2)
void plotrmove(long ne, GEN x, GEN y)
void plotrpoint(long ne, GEN x, GEN y)
void plotscale(long ne, GEN x1, GEN x2, GEN y1, GEN y2)
void plotstring(long ne, char *x, long dir)

```

18.0.7 Obsolete functions. These draw directly to a PostScript file specified by a global variable and should no longer be used. Use `plotexport` and friends instead.

```

void psdraw(GEN list, long flag)
GEN psplothraw(GEN listx, GEN listy, long flag)
GEN psploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long
prec) draw to a PostScript file.

```

18.0.8 Dump rectwindows to a PostScript or SVG file.

w, x, y are three `t_VECSMALLs` indicating the rectwindows to dump, at which offsets. If T is `NULL`, rescale with respect to the installed graphic engine dimensions; else with respect to T .

```

char* rect2ps(GEN w, GEN x, GEN y, PARI_plot *T)
char* rect2ps_i(GEN w, GEN x, GEN y, PARI_plot *T, int plotps) if plotps is 0, as above;
else private version used to implement the plotps graphic engine (do not rescale, rotate to portrait
orientation).
char* rect2svg(GEN w, GEN x, GEN y, PARI_plot *T)

```

18.0.9 Technical functions exported for convenience.

```

void pari_plot_by_file(const char *env, const char *suf, const char *img) backend
used by the plotps and plotsvg graphic engines.
void colorname_to_rgb(const char *s, int *r, int *g, int *b) convert an X11 colorname
to RGB values.
void color_to_rgb(GEN c, int *r, int *g, int *b) convert a pari color (t_VECSMALL RGB
triple or t_STR name) to RGB values.
void long_to_rgb(long c, int *r, int *g, int *b) split a standard hexadecimal color value
0xfdf5e6 to its rgb components (0xfd, 0xf5, 0xe6).

```

Appendix A: A Sample program and Makefile

We assume that you have installed the PARI library and include files as explained in Appendix A or in the installation guide. If you chose differently any of the directory names, change them accordingly in the Makefiles.

If the program example that we have given is in the file `extgcd.c`, then a sample Makefile might look as follows. Note that the actual file `examples/Makefile` is more elaborate and you should have a look at it if you intend to use `install()` on custom made functions.

```
CC = cc
INCDIR = /usr/include
LIBDIR = /usr/libx32
CFLAGS = -O -I$(INCDIR) -L$(LIBDIR)

all: extgcd

extgcd: extgcd.c
    $(CC) $(CFLAGS) -o extgcd extgcd.c -lpari -lm
```

We then give the listing of the program `examples/extgcd.c` seen in detail in Section [4.10](#).

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
    pari_sp av = avma;
    GEN ux = gen_1, vx = gen_0, a = A, b = B;
    if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
    if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
    if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
    }
    *U = ux;
    *V = diviixact( subii(a, mulii(A,ux)), B );
    gerepileall(av, 3, &a, U, V); return a;
}

int
```

```
main()
{
  GEN x, y, d, u, v;
  pari_init(1000000,2);
  printf("x = "); x = gp_read_stream(stdin);
  printf("y = "); y = gp_read_stream(stdin);
  d = extgcd(x, y, &u, &v);
  pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
  pari_close();
  return 0;
}
```

Appendix B: PARI and threads

To use PARI in multi-threaded programs, you must configure it using `Configure --enable-tls`. Your system must implement the `__thread` storage class. As a major side effect, this breaks the `libpari` ABI: the resulting library is not compatible with the old one, and `-tls` is appended to the PARI library `soname`. On the other hand, this library is now thread-safe.

PARI provides some functions to set up PARI subthreads. In our model, each concurrent thread needs its own PARI stack. The following scheme is used:

Child thread:

```
void *child_thread(void *arg)
{
    GEN data = pari_thread_start((struct pari_thread*)arg);
    GEN result = ...; /* Compute result from data */
    pari_thread_close();
    return (void*)result;
}
```

Parent thread:

```
pthread_t th;
struct pari_thread pth;
GEN data, result;

pari_thread_alloc(&pth, s, data);
pari_thread_sync();
pthread_create(&th, NULL, &child_thread, (void*)&pth); /* start child */
... /* do stuff in parent */
pthread_join(th, (void*)&result); /* wait until child terminates */
result = gcopy(result); /* copy result from thread stack to main stack */
pari_thread_free(&pth); /* ... and clean up */
```

`void pari_thread_valloc(struct pari_thread *pth, size_t s, size_t v, GEN arg)` Allocate a PARI stack of size `s` which can grow to at most `v` (as with `parisize` and `parisizemax`) and associate it, together with the argument `arg`, with the PARI thread data `pth`.

`void pari_thread_alloc(struct pari_thread *pth, size_t s, GEN arg)` As above but the stack cannot grow beyond `s`.

`void pari_thread_free(struct pari_thread *pth)` Free the PARI stack attached to the PARI thread data `pth`. This is called after the child thread terminates, i.e. after `pthread_join` in the parent. Any GEN objects returned by the child in the thread stack need to be saved before running this command.

`void pari_thread_sync(void)` Record states from the main thread so that they are available to `pari_thread_start()`. Must be called in the main thread before the subthreads starts.

`void pari_thread_init(void)` Initialize the thread-local PARI data structures. This function is called by `pari_thread_start`.

`GEN pari_thread_start(struct pari_thread *t)` Initialize the thread-local PARI data structures and set up the thread stack using the PARI thread data `pth`. This function returns the thread argument `arg` that was given to `pari_thread_alloc`.

`void pari_thread_close(void)` Free the thread-local PARI data structures, but keeping the thread stack, so that a `GEN` returned by the thread remains valid.

Under this model, some PARI states are reset in new threads. In particular

- the random number generator is reset to the starting seed;
- the system stack exhaustion checking code, meant to catch infinite recursions, is disabled (use `pari_stackcheck_init()` to reenable it);
- cached real constants (returned by `mppi`, `mpeuler` and `mplog2`) are not shared between threads and will be recomputed as needed;

The following sample program can be compiled using

```
cc thread.c -o thread.o -lpari -lpthread
```

(Add `-I/-L` paths as necessary.)

```
#include <pari/pari.h> /* Include PARI headers */
#include <pthread.h> /* Include POSIX threads headers */

void *
mydet(void *arg)
{
    GEN F, M;
    /* Set up thread stack and get thread parameter */
    M = pari_thread_start((struct pari_thread*) arg);
    F = det(M);
    /* Free memory used by the thread */
    pari_thread_close();
    return (void*)F;
}

void *
myfactor(void *arg) /* same principle */
{
    GEN F, N;
    N = pari_thread_start((struct pari_thread*) arg);
    F = factor(N);
    pari_thread_close();
    return (void*)F;
}

int
main(void)
{
    GEN M,N1,N2, F1,F2,D;

```

```

pthread_t th1, th2, th3; /* POSIX-thread variables */
struct pari_thread pth1, pth2, pth3; /* pari thread variables */

/* Initialise the main PARI stack and global objects (gen_0, etc.) */
pari_init(4000000,500000);
/* Compute in the main PARI stack */
N1 = addis(int2n(256), 1); /* 2^256 + 1 */
N2 = subis(int2n(193), 1); /* 2^193 - 1 */
M = mathilbert(80);
/* Sync with main thread */
pari_thread_sync();
/* Allocate pari thread structures */
pari_thread_alloc(&pth1,4000000,N1);
pari_thread_alloc(&pth2,4000000,N2);
pari_thread_alloc(&pth3,4000000,M);
/* pthread_create() and pthread_join() are standard POSIX-thread
 * functions to start and get the result of threads. */
pthread_create(&th1,NULL, &myfactor, (void*)&pth1);
pthread_create(&th2,NULL, &myfactor, (void*)&pth2);
pthread_create(&th3,NULL, &mydet, (void*)&pth3); /* Start 3 threads */
pthread_join(th1,(void*)&F1);
pthread_join(th2,(void*)&F2);
pthread_join(th3,(void*)&D); /* Wait for termination, get the results */
pari_printf("F1=%Ps\nF2=%Ps\nlog(D)=%Ps\n", F1, F2, glog(D,3));
pari_thread_free(&pth1);
pari_thread_free(&pth2);
pari_thread_free(&pth3); /* clean up */
return 0;
}

```

Index

SomeWord refers to PARI-GP concepts.
SomeWord is a PARI-GP keyword.
SomeWord is a generic index entry.

A

ABC_to_bnr	291
abelian_group	235
abgrp_get_cyc	268
abgrp_get_gen	268
abgrp_get_no	268
abscmpii	89
abscmpiu	89
abscmprr	89
abscmpui	89
absdiviu_rem	93
absequalii	89
absequaliu	89
absequalui	89
absfrac	223
absfrac_shallow	223
absi	87
absi_shallow	87
absr	87
absrnz_equal1	89
absrnz_equal2n	89
abstorel	292
absZ_factor	158
absZ_factor_limit	158
addhelp	74
addii	15
addii_sign	91
addir	15
addir_sign	91
addis	15
addiu	90
addll	77
addllx	77
addmul	77
addmulii	91
addmulii_inplace	91
addmuliu	91
addmuliu_inplace	91
addri	15
addr	15
addr_sign	92
addsi_sign	92
addui	90
addui_sign	92
addumului	91
adduu	90
affc_fixlg	232
affects_sign	60
affects_sign_safe	60
affgr	82
affii	82
affir	82
affiz	82
affrr	82
affrr_fixlg	82, 232
affsi	82
affsr	82
affsz	82
affui	82
affur	82
alarm	247
alginit	308
alglat_get_primbasis	309
alglat_get_scalar	309
algsimpledec_ss	309
algtype	308
alg_complete	308
alg_csa_table	307
alg_cyclic	307
alg_get_absdim	308
alg_get_abssplitting	309
alg_get_aut	308
alg_get_auts	308
alg_get_b	309
alg_get_basis	309
alg_get_center	309
alg_get_char	309
alg_get_degree	308
alg_get_dim	308
alg_get_hasse_f	309
alg_get_hasse_i	309
alg_get_invbasis	309
alg_get_multable	309
alg_get_relmultable	309
alg_get_splitpol	309
alg_get_splittingbasis	309
alg_get_splittingbasisinv	309
alg_get_splittingdata	309
alg_get_splittingfield	309
alg_get_tracebasis	309
alg_hasse	307
alg_hilbert	308
alg_matrix	308

alg_model	308
alg_type	308
assignment	26
avma	17, 26

B

bb_algebra	195
bb_field	193
bb_group	191
bb_ring	196
bernfrac	233, 234
Bernoulli	233, 234
bernreal	233, 234
bezout	46, 96
bfffo	77
bid_get_arch	271
bid_get_archp	271
bid_get_cyc	272
bid_get_fact	271
bid_get_gen	272
bid_get_gen_nocheck	272
bid_get_grp	271
bid_get_ideal	271
bid_get_mod	271
bid_get_no	271
bid_get_sarch	272
bid_get_sprk	272
bid_get_U	272
BIGDEFAULTPREC	16, 62
bigomegau	99
BIL	49
binary_2k	85, 86
binary_2k_nv	86
binary_zv	85
bincopy_relink	65
binomial	223
binomialuu	223
bin_copy	65
BITS_IN_HALFULONG	62
BITS_IN_LONG	16, 49, 62, 86
bits_to_int	86
bits_to_u	86
bit_accuracy	16, 57
bit_accuracy_mul	57
bit_prec	57
bl_base	69
bl_next	69
bl_num	69

bl_prev	69
bl_refc	69
bnfgwgeneric	292
bnfisprincipal0	273, 286, 289
bnfisunit	278
bnfnewprec	273, 287
bnfnewprec_shallow	273
bnfttestprimes	288
bnf_build_cycgen	272
bnf_build_matalpha	272
bnf_build_units	273
bnf_get_clgp	270
bnf_get_cyc	270
bnf_get_fu	270
bnf_get_fu_nocheck	270
bnf_get_gen	270
bnf_get_logfu	270
bnf_get_nf	269
bnf_get_no	270
bnf_get_reg	270
bnf_get_tuN	270
bnf_get_tuU	270
bnrchar_primitive	291
bnrclassno	291
bnrconductor	291
bnrconductor_i	291
bnrdisc	291
bnrdisclist0	294
bnrisconductor	291
bnrisprincipal	294
bnrnewprec	273
bnrnewprec_shallow	273
bnrsurjection	291
bnr_get_bid	270
bnr_get_bnf	270
bnr_get_clgp	270
bnr_get_cyc	270
bnr_get_gen	270
bnr_get_gen_nocheck	270
bnr_get_mod	270
bnr_get_nf	270
bnr_get_no	270
both_odd	78
boundfact	158
BPSW_isprime	163
BPSW_psp	163
brent_kung_optpow	195
brute	242
buchimag	294

chinese1_coprime_Z	146	cmp_universal	174, 210, 214
chk_gerepileupto	68	colorname_to_rgb	336
classno	296	colors	241, 242
classno2	296	color_to_rgb	336
clcm	96	coltrunc_init	56
cleanroots	181, 225	column vector	33
clean_Z_factor	161	col_ei	204
<i>clone</i>	68	compile_str	54, 260
clone	14, 27	complex number	31
CLONEBIT	62	compo	61
closemodinvertible	305	conjclasses_algcenter	309
<i>closure</i>	72	conjclasses_repr	236
closure	34	conjvec	225, 232
closure_arity	34	conj_i	223
closure_callgen1	72, 259	constant_coeff	32, 61
closure_callgen1prec	259	constcatalan	233
closure_callgen2	259	consteuler	233
closure_callgenall	259	constlog2	233
closure_callgenvec	259	constpi	233
closure_callgenvecprec	259	const_col	255
closure_callvoid1	259	const_F2v	113
closure_context	260	const_vec	255
closure_deriv	260	const_vecsmall	255
closure_disassemble	259	content	216
closure_err	261	conversions	27
closure_evalbrk	259	coprimes_zv	96
closure_evalgen	72, 259	copy	27
closure_evalnobrk	259	copybin_unlink	65
closure_evalres	259	copyifstack	68
closure_evalvoid	72, 259	copy_bin	65
closure_func_err	53	copy_bin_canon	65
closure_is_variadic	34	core	162
closure_trapgen	260	core2	162
cmpii	88	core2partial	162
cmpir	88	corediscs	100
cmpis	88	corepartial	162
cmpiu	88	coreu	100
cmpri	88	coreu_fact	100
cmprr	88	cornacchia	297
cmprs	88	cornacchia2	298
cmpsi	88	cornacchia2_sqrt	298
cmpsr	88	creation	24
cmpui	88	cvstop2	200
cmp_Flx	214	cvtop	200
cmp_nodata	214	cvtop2	200
cmp_padic	214	cxcompotor	199
cmp_prime_ideal	214	cxEk	232
cmp_prime_over_p	214	cxexpm1	223
cmp_RgX	214	cxnorm	223

cxredsl2	232	degpol	32, 58, 225
cxredsl2_i	232	degree	32, 224
cxtofp	199	delete_var	36, 70
cxtoreal	200	denom_i	216
cx_approx_equal	223	deplin	304
cyc2elts	306	derivser	226
cyclicgroup	235	destruction	18
cyclic_perm	234	detint	167
cyc_normalize	305	diagonal_shallow	208
cyc_pow	234	dicyclicgroup	235
cyc_pow_perm	234	diffptr	14
c_ERR	241	discrayabslist	294
c_HELP	242	discrayabslistarch	294
c_HIST	241	disc_best_modinv	324
C_INCLUDE_PATH	13	diviexact	92
c_INPUT	242	diviiround	84
c_NONE	242	divisors	42
c_OUTPUT	242	divisorsu	100
c_PROMPT	242	divisorsu_fact	100
c_TIME	242	divis_rem	93

D

datadir	76	divll	77
dbgGEN	40	divll_pre	79
dbg_gerepile	68	divsBIL	62
dbg_gerepileupto	68	divsi_rem	93
dbg_pari_heap	41	divss_rem	93
dblcoro526	328	dvdi	92
dblexpo	199	dvdiiz	92
dbllambertW0	328	dvdis	92
dbllambertW_1	328	dvdisz	92
dbllemma526	328	dvdiu	92
dbllog2r	199	dvdiuz	92
dblmantissa	199	dvdsi	92
dblmodulus	199	dvdui	92
dbltor	27, 199	dvmdii	92
debug	40	dvmdiiz	93
debugging	40	dvmdis	93
DEBUGLEVEL	40	dvmdsBIL	62
DEBUGMEM	40	dvmdsi	93
debugmem	40	dvmdss	93
default0	52	dvmduBIL	62
DEFAULTPREC	16, 62	dynamic array	251
definite binary quadratic form	33	d_ACKNOWLEDGE	261, 262, 264
deg1pol	203	d_INITRC	261, 263
deg1pol_shallow	208	d_RETURN	261, 263, 264
deg1_from_roots	209	d_SILENT	262
deg2pol_shallow	208		

E

echo	55	ellQp_AGM	314
ecpp	163	ellQp_Ei	233
ecppexport	163	ellQp_get_p	314
ecppisvalid	163	ellQp_get_prec	314
ec_2divpol_evalx	316	ellQp_get_zero	314
ec_3divpol_evalx	316	ellQp_L	314
ec_bmodel	316	ellQp_q	314
ec_dFdx_evalQ	316	ellQp_root	314
ec_dFdy_evalQ	316	ellQp_Tate_uniformization	314
ec_dmFdy_evalQ	316	ellQp_u	314
ec_f_evalx	316	ellQp_u2	314
ec_half_deriv_2divpol_evalx	316	ellQ_get_CM	312, 319
ec_h_evalx	316	ellQ_get_N	312
effective length	29	ellQ_get_Nfa	312
ei_multable	276	ellrootno_global	313
ellff_get_o	315	ellR_ab	315
ell	311	ellR_area	315
ellanalyticrank	328	ellR_eta	315
ellanal_globalred	313	ellR_get_prec	315
ellanal_globalred_all	313	ellR_get_sign	316
ellap	316	ellR_omega	315
ellap_CM_fast	312	ellR_roots	315
ellbasechar	316	elltors	314
ellchangeinvert	316	elltors0	313
elldatagenerators	313	elltrace_extension	317
ellff_get_a4a6	315	ell_get_a1	312
ellff_get_card	314	ell_get_a2	312
ellff_get_D	315	ell_get_a3	312
ellff_get_field	314	ell_get_a4	312
ellff_get_gens	314	ell_get_a6	312
ellff_get_group	314	ell_get_b2	312
ellff_get_m	314	ell_get_b4	312
ellff_get_o	315	ell_get_b6	312
ellff_get_p	314	ell_get_b8	312
ellheightoo	313	ell_get_c4	312
ellinf	316	ell_get_c6	312
ellinit	311	ell_get_disc	312
ellintegralmodel	313	ell_get_j	312
ellintegralmodel_i	313	ell_get_type	311, 317
ellisogeny	312	ell_is_inf	316
ellL1	328	ell_is_integral	312
ellnf_get_bnf	314	eltabstorel	292
ellnf_get_nf	314	eltabstorel_lift	292
ellnf_vecarea	314	eltreltoabs	292
ellnf_veceta	314	embednorm_T2	288
ellnf_vecomega	314	embed_disc	288
ellorder	313	embed_norm	288
ellorder_Q	313	embed_roots	288
ellQp_ab	314	embed_T2	288

F2xqM_F2xqC_gauss	141	F2xq_ellgroup	320
F2xqM_F2xqC_invimage	141	F2xq_elltwist	321
F2xqM_F2xqC_mul	141	F2xq_inv	140
F2xqM_gauss	141	F2xq_invsafe	140
F2xqM_image	141	F2xq_log	141
F2xqM_indexrank	141	F2xq_matrix_pow	141
F2xqM_inv	141	F2xq_mul	140
F2xqM_invimage	141	F2xq_order	141
F2xqM_ker	142	F2xq_pow	140
F2xqM_mul	142	F2xq_powers	141
F2xqM_rank	142	F2xq_powu	140
F2xqM_suppl	142	F2xq_pow_init	140
F2xqXQV_red	144	F2xq_pow_table	140
F2xqXQ_autpow	144	F2xq_sqr	140
F2xqXQ_auttrace	144	F2xq_sqrt	141
F2xqXQ_inv	143	F2xq_sqrtn	141
F2xqXQ_invsafe	143	F2xq_sqrt_fast	141
F2xqXQ_mul	144	F2xq_trace	140
F2xqXQ_pow	144	F2xV_to_F2m	157
F2xqXQ_powers	144	F2xV_to_FlxV_inplace	155
F2xqXQ_sqr	144	F2xV_to_ZXV_inplace	155
F2xqX_ddf	144	F2xXC_to_ZXXC	142
F2xqX_degfact	144	F2xX_add	142
F2xqX_div	143	F2xX_deriv	142
F2xqX_divrem	143	F2xX_F2x_add	142
F2xqX_extgcd	143	F2xX_F2x_mul	142
F2xqX_F2xqXQV_eval	143	F2xX_renormalize	142
F2xqX_F2xqXQ_eval	143	F2xX_to_Kronecker	142
F2xqX_F2xq_mul	143	F2xX_to_ZXX	142
F2xqX_F2xq_mul_to_monic	143	F2xY_degrees	142
F2xqX_factor	143	F2xY_F2xqV_evalx	142
F2xqX_factor_squarefree	143	F2xY_F2xq_evalx	142
F2xqX_gcd	143	F2x_1_add	139
F2xqX_get_red	143	F2x_add	139
F2xqX_invBarrett	143	F2x_clear	139
F2xqX_isplayer	143	F2x_coeff	139
F2xqX_mul	143	F2x_copy	139
F2xqX_normalize	143	F2x_ddf	140
F2xqX_powu	143	F2x_deflate	140
F2xqX_red	143	F2x_degfact	140
F2xqX_rem	143	F2x_degree	139
F2xqX_roots	143	F2x_deriv	140
F2xqX_sqr	143	F2x_div	140
F2xq_Artin_Schreier	141	F2x_divrem	139
F2xq_autpow	141	F2x_equal	139
F2xq_conjvec	141	F2x_equal1	139
F2xq_div	140	F2x_eval	140
F2xq_ellcard	320	F2x_even_odd	140
F2xq_ellgens	320	F2x_extgcd	140

F2x_F2xqV_eval	141	famat_pow	277
F2x_F2xq_eval	141	famat_pows_shallow	277
F2x_factor	140	famat_pow_shallow	277
F2x_factor_squarefree	140	famat_reduce	277, 278
F2x_flip	139	famat_sqr	277
F2x_Frobenius	140	famat_to_nf	278
F2x_gcd	140	famat_to_nf_moddivisor	290
F2x_get_red	138	famat_to_nf_modideal_coprime	290
F2x_halfgcd	140	famat_Z_gcd	277
F2x_issquare	140	fetch_user_var	35, 70
F2x_is_irred	140	fetch_var	36, 70
F2x_matFrobenius	140	fetch_var_higher	36
F2x_mul	139	fetch_var_value	36, 70
F2x_rem	140	FFM_deplin	230
F2x_renormalize	140	FFM_det	230
F2x_set	139	FFM_FFC_gauss	229
F2x_shift	140	FFM_FFC_invimage	229
F2x_sqr	139	FFM_FFC_mul	229
F2x_sqrt	140	FFM_gauss	230
F2x_to_F2v	157	FFM_image	230
F2x_to_F2xX	139	FFM_indexrank	230
F2x_to_Flx	139	FFM_inv	230
F2x_to_ZX	139	FFM_invimage	230
F2x_valrem	140	FFM_ker	230
factmod	145	FFM_mul	230
factor	311	FFM_rank	230
factorback	220	FFM_suppl	230
factoredpolred	294	FFXQ_inv	230
factoredpolred2	294	FFXQ_mul	230
factorial_lval	87	FFXQ_sqr	230
factorint	160	FFX_ddf	229
factoru	159	FFX_degfact	229
factoru_pow	159	FFX_factor	229
factor_Aurifeuille	159	FFX_factor_squarefree	229
factor_Aurifeuille_prime	159	FFX_isplayer	229
factor_pn_1	158	FFX_mul	229
factor_pn_1_limit	159	FFX_preimage	229
factor_proven	162	FFX_rem	229
famat	277	FFX_roots	229
famat_small_reduce	278	FFX_sqr	229
famat_div_shallow	277	FF_1	227
famat_inv	277	FF_add	227
famat_inv_shallow	277	FF_charpoly	228
famat_makecoprime	290	FF_conjvec	228
famat_mul	277	FF_div	227
famat_mulpow	277	FF_ellcard	317
famat_mulpows_shallow	277	FF_ellcard_SEA	317
famat_mulpow_shallow	277	FF_ellgens	317
famat_mul_shallow	277	FF_ellgroup	317

FF_elllog	317	FF_Z_Z_muldiv	228
FF_ellmul	317	file_is_binary	242
FF_ellorder	317	finite field element	31
FF_ellrandom	317	fixlg	67, 83
FF_elltatepairing	317	Flc_Flv_mul	110
FF_elltwist	317	Flc_FpV_mul	111
FF_ellweilpairing	317	Flc_lincomb1_inplace	110
FF_equal	227	Flc_to_mod	144
FF_equal0	227	Flc_to_ZC	155
FF_equal1	227	Flc_to_ZC_inplace	155
FF_equality1	227	Fle_add	319
FF_f	226	Fle_changepoint	320
FF_Frobenius	228	Fle_changepointinv	320
FF_gen	226	Fle_dbl	319
FF_inv	228	Fle_log	319
FF_ispower	228	Fle_mul	319
FF_issquare	228	Fle_mulu	319
FF_issquareall	228	Fle_order	319
FF_log	228	Fle_sub	319
FF_map	228	Fle_to_Flj	320
FF_minpoly	228	Flj_add_pre	320
FF_mod	226	Flj_dbl_pre	320
FF_mul	227	Flj_mulu_pre	320
FF_mul2n	228	Flj_neg	320
FF_neg	228	Flj_to_Fle_pre	320
FF_neg_i	228	Flm_add	111
FF_norm	228	Flm_adjoint	111
FF_order	228	Flm_center	109
FF_p	226	Flm_charpoly	111
FF_pow	228	Flm_copy	109
FF_primroot	228	Flm_deplin	111
FF_p_i	226	Flm_det	111
FF_q	226	Flm_det_sp	111
FF_Q_add	227	Flm_Flc_gauss	111
FF_samefield	227	Flm_Flc_invimage	111
FF_sqr	228	Flm_Flc_mul	110
FF_sqrt	228	Flm_Flc_mul_pre	110
FF_sqrtn	228	Flm_Flc_mul_pre_Flx	110
FF_sub	227	Flm_Fl_add	110
FF_to_F2xq	226	Flm_Fl_mul	110
FF_to_F2xq_i	226	Flm_Fl_mul_inplace	110
FF_to_Flxq	226	Flm_gauss	111
FF_to_Flxq_i	226	Flm_hess	112
FF_to_FpXQ	226	Flm_image	112
FF_to_FpXQ_i	226	Flm_indexrank	111
FF_trace	228	Flm_intersect	112
FF_zero	227	Flm_inv	111
FF_Z_add	227	Flm_invimage	111
FF_Z_mul	227	Flm_ker	112

Flm_ker_sp	112	FlxC_neg	133
Flm_mul	111	FlxC_sub	133
Flm_neg	110	FlxC_to_F2xC	157
Flm_powers	111	FlxC_to_ZXC	155
Flm_powu	111	FlxM_eval_powers_pre	133
Flm_rank	112	FlxM_Flx_add_shallow	113
Flm_row	111	FlxM_neg	133
Flm_sub	111	FlxM_sub	133
Flm_suppl	112	FlxM_to_ZXM	155
Flm_to_F2m	113	Flxn_inv	130
Flm_to_FlxV	156	Flxn_mul	130
Flm_to_FlxX	156	FlxqC_Flxq_mul	114
Flm_to_mod	144	FlxqE_add	322
Flm_to_ZM	155	FlxqE_changepoint	322
Flm_to_ZM_inplace	155	FlxqE_changepointinv	322
Flm_transpose	112	FlxqE_dbl	322
floorr	84	FlxqE_log	322
floor_safe	84	FlxqE_mul	322
flush	240	FlxqE_neg	322
Flv_add	110	FlxqE_order	322
Flv_add_inplace	110, 284	FlxqE_sub	322
Flv_center	109	FlxqE_tatepairing	322
Flv_copy	109	FlxqE_weilpairing	322
Flv_dotproduct	110	FlxqM_deplin	114
Flv_dotproduct_pre	110	FlxqM_det	114
Flv_Flm_polint	133	FlxqM_FlxqC_gauss	114
Flv_Fl_div	110	FlxqM_FlxqC_invimage	114
Flv_Fl_div_inplace	110	FlxqM_FlxqC_mul	114
Flv_Fl_mul	110	FlxqM_Flxq_mul	114
Flv_Fl_mul_inplace	110	FlxqM_gauss	114
Flv_Fl_mul_part_inplace	110	FlxqM_image	114
Flv_inv	111	FlxqM_indexrank	114
Flv_invVandermonde	133	FlxqM_inv	114
Flv_inv_inplace	111	FlxqM_invimage	114
Flv_inv_pre	111	FlxqM_ker	114
Flv_inv_pre_inplace	111	FlxqM_mul	114
Flv_neg	110	FlxqM_rank	114
Flv_neg_inplace	110	FlxqM_suppl	114
Flv_polint	133	FlxqV_dotproduct	113
Flv_prod	110	FlxqV_roots_to_pol	134
Flv_prod_pre	111	FlxqXQ_autpow	138
Flv_roots_to_pol	133	FlxqXQ_autsum	138
Flv_sub	110	FlxqXQ_auttrace	138
Flv_sub_inplace	110	FlxqXQ_div	138
Flv_sum	110	FlxqXQ_halfFrobenius	138
Flv_to_F2v	113	FlxqXQ_inv	138, 144
Flv_to_Flx	156	FlxqXQ_invsafe	138, 144
Flv_to_ZV	155	FlxqXQ_matrix_pow	138
FlxC_eval_powers_pre	133	FlxqXQ_minpoly	138

FlxqXQ_mul	138	Flxq_elltwist	322
FlxqXQ_pow	138	Flxq_ffisom_inv	134
FlxqXQ_powers	138	Flxq_inv	134
FlxqXQ_powu	138	Flxq_invsafe	134
FlxqXQ_sqr	138	Flxq_is2npower	134
FlxqXV_prod	137	Flxq_issquare	134
FlxqX_ddf	138	Flxq_log	134
FlxqX_ddf_degree	138	Flxq_lroot	135
FlxqX_degfact	138	Flxq_lroot_fast	135
FlxqX_div	137	Flxq_matrix_pow	134
FlxqX_divrem	137	Flxq_minpoly	135
FlxqX_dotproduct	137	Flxq_mul	134
FlxqX_extgcd	137	Flxq_norm	135
FlxqX_factor	137	Flxq_order	134
FlxqX_factor_squarefree	138	Flxq_pow	134
FlxqX_FlxqXQV_eval	138	Flxq_powers	134
FlxqX_FlxqXQ_eval	138	Flxq_powu	134
FlxqX_Flxq_mul	137	Flxq_pow_init	134
FlxqX_Flxq_mul_to_monic	137	Flxq_pow_table	134
FlxqX_Frobenius	137	Flxq_sqr	134
FlxqX_gcd	137	Flxq_sqrt	134
FlxqX_get_red	136	Flxq_sqrtn	134
FlxqX_halfgcd	137	Flxq_sub	134
FlxqX_invBarrett	137	Flxq_trace	135
FlxqX_ispower	137	FlxT_red	134
FlxqX_is_squarefree	137	FlxV_Flc_mul	133
FlxqX_mul	137	FlxV_Flx_fromdigits	130
FlxqX_nbfact	138	FlxV_prod	133
FlxqX_nbfact_Frobenius	138	FlxV_red	133
FlxqX_nbroots	138	FlxV_to_Flm	156
FlxqX_normalize	137	FlxV_to_ZXV	155
FlxqX_powu	137	FlxV_to_ZXV_inplace	155
FlxqX_red	137	FlxXC_to_F2xXC	142
FlxqX_rem	137	FlxXC_to_ZXXC	155
FlxqX_roots	137	FlxXM_to_ZXXM	155
FlxqX_safegcd	137	FlxXV_to_FlxM	157
FlxqX_sqr	137	FlxX_add	135
Flxq_add	134	FlxX_deriv	135
Flxq_autpow	134	FlxX_double	135
Flxq_autsum	134	FlxX_Flx_add	135
Flxq_auttrace	134	FlxX_Flx_mul	135
Flxq_charpoly	135	FlxX_Flx_sub	135
Flxq_conjvec	135	FlxX_Fl_mul	135
Flxq_div	134	FlxX_neg	135
Flxq_ellcard	322	FlxX_renormalize	136
Flxq_ellgens	322	FlxX_resultant	136
Flxq_ellgroup	322	FlxX_shift	136
Flxq_ellj	321	FlxX_sub	135
Flxq_ellj_to_a4a6	321	FlxX_swap	136

FlxX_to_F2xX	142	Flx_gcd	131
FlxX_to_Flm	156	Flx_get_red	129
FlxX_to_Flx	156	Flx_halfgcd	131
FlxX_to_FlxC	156	Flx_halve	130
FlxX_to_ZXX	155	Flx_inflate	132
FlxX_triple	135	Flx_invBarrett	132
FlxYqq_pow	136	Flx_ispower	131
FlxY_degrees	135	Flx_is_irred	132
FlxY_evalx	135	Flx_is_smooth	132
FlxY_evalx_powers_pre	135	Flx_is_squarefree	132
FlxY_eval_powers_pre	136	Flx_lead	130
FlxY_FlxqV_evalx	136	Flx_matFrobenius	131
FlxY_Flxq_evalx	136	Flx_mod_Xn1	131
FlxY_Flx_div	135	Flx_mod_Xnm1	131
FlxY_Flx_translate	135	Flx_mul	130
Flx_add	130	Flx_mulu	130
Flx_constant	130	Flx_nbfact	133
Flx_copy	130	Flx_nbfact_by_degree	133
Flx_ddf	131	Flx_nbfact_Frobenius	133
Flx_deflate	132	Flx_nbroots	133
Flx_degfact	131, 133	Flx_neg	130
Flx_deriv	130	Flx_neg_inplace	130
Flx_diff1	130	Flx_normalize	131
Flx_digits	130	Flx_oneroot	131
Flx_div	130	Flx_oneroot_split	131
Flx_divrem	130	Flx_powu	130
Flx_div_by_X_x	132	Flx_recip	132
Flx_dotproduct	132	Flx_red	130
Flx_double	130	Flx_rem	130
Flx_equal	130	Flx_renormalize	132
Flx_equal1	130	Flx_rescale	131
Flx_eval	132	Flx_resultant	132
Flx_eval_powers_pre	132	Flx_roots	131
Flx_eval_pre	132	Flx_rootsff	131
Flx_extgcd	131	Flx_shift	132
Flx_extresultant	132	Flx_splitting	132
Flx_factor	131	Flx_sqr	130
Flx_factorff_irred	131	Flx_sub	130
Flx_factor_squarefree	131	Flx_to_F2x	139
Flx_ffintersect	133	Flx_to_Flv	156
Flx_ffisom	131	Flx_to_FlxX	155
Flx_Flv_multieval	132	Flx_to_ZX	155
Flx_FlxqV_eval	134	Flx_to_ZX_inplace	155
Flx_Flxq_eval	134	Flx_translate1	130
Flx_FlxY_resultant	136	Flx_triple	130
Flx_Fl_add	130	Flx_val	132
Flx_Fl_mul	130	Flx_valrem	132
Flx_Fl_mul_to_monic	130	Fly_to_FlxY	157
Flx_Frobenius	131	Fl_2gener_pre	80

Fl_add	78	format	40
Fl_addmulmul_pre	80	forpart	42
Fl_addmul_pre	79	forpart_init	42
Fl_center	78	forpart_next	42
Fl_div	79	forpart_prev	42
Fl_double	78	forpart_t	42
Fl_elldisc	319	forperm	43
Fl_elldisc_pre	319	forperm_init	43
Fl_ellj	319	forperm_next	43
Fl_ellj_pre	319	forprime	42
Fl_ellj_to_a4a6	319	forprimestep_init	163
Fl_elltrace	319	forprime_init	43, 44, 163, 164
Fl_elltrace_CM	319	forprime_next	43, 163
Fl_elltwist	319	forprime_t	43, 44
Fl_elltwist_disc	319	forsubgroup	42
Fl_halve	78	forsubgroup(H = G, B,)	42
Fl_inv	78	forsubset	43
Fl_invgen	78	forsubset_init	43
Fl_invsafe	78	forsubset_next	43
Fl_log	79	forvec	42
Fl_log_pre	80	forvec_init	42
Fl_mul	78	forvec_next	42
Fl_mul_pre	79	FpC_add	107
Fl_neg	78	FpC_center	107
Fl_order	79	FpC_center_inplace	107
Fl_powers	79	FpC_FpV_mul	108
Fl_powers_pre	80	FpC_Fp_mul	108
Fl_powu	79	FpC_ratlift	149
Fl_powu_pre	80	FpC_red	107
Fl_sqr	78	FpC_sub	107
Fl_sqrt	79	FpC_to_mod	144
Fl_sqrtl	79	FpE_add	318
Fl_sqrtl_pre	80	FpE_changepoint	318
Fl_sqrtn	79	FpE_changepointinv	318
Fl_sqrtn_pre	80	FpE_dbl	318
Fl_sqrt_pre	80	FpE_log	319
Fl_sqrt_pre_i	80	FpE_mul	318
Fl_sqr_pre	79	FpE_neg	318
Fl_sub	78	FpE_order	318
Fl_to_Flx	156	FpE_sub	318
Fl_triple	78	FpE_tatepairing	319
forallsubset_init	43	FpE_to_FpJ	320
forcomposite	42	FpE_to_mod	319
forcomposite_init	42	FpE_weilpairing	319
forcomposite_next	42	FpJ_add	320
fordiv	42	FpJ_dbl	320
forell	42	FpJ_mul	320
forell(ell,a,b,)	42	FpJ_neg	320
forksubset_init	43	FpJ_to_FpE	320

FpMs_FpCs_solve	175	FpXM_FpXQV_eval	122
FpMs_FpCs_solve_safe	175	FpXn_exp	122
FpMs_FpC_mul	175	FpXn_inv	122
FpMs_leftkernel_elt	175	FpXn_mul	122
FpM_add	107	FpXn_sqr	122
FpM_center	107	FpXQC_to_mod	144
FpM_center_inplace	107	FpXQE_add	323
FpM_charpoly	109	FpXQE_changepoint	323
FpM_deplin	108	FpXQE_changepointinv	323
FpM_det	108	FpXQE_dbl	323
FpM_FpC_gauss	108	FpXQE_log	323
FpM_FpC_invimage	108	FpXQE_mul	323
FpM_FpC_mul	108	FpXQE_neg	323
FpM_FpC_mul_FpX	108	FpXQE_order	323
FpM_Fp_mul	108	FpXQE_sub	323
FpM_gauss	108	FpXQE_tatepairing	323
FpM_hess	109	FpXQE_weilpairing	323
FpM_image	108	FpXQM_autsum	122
FpM_indexrank	109	FpXQXn_exp	126
FpM_intersect	108	FpXQXn_inv	126
FpM_inv	108	FpXQXn_mul	126
FpM_invimage	108	FpXQXn_sqr	125
FpM_ker	108	FpXQXQ_autpow	127
FpM_mul	108	FpXQXQ_autsum	127
FpM_powu	108	FpXQXQ_auttrace	127
FpM_rank	108	FpXQXQ_div	126
FpM_ratlift	149	FpXQXQ_halfFrobenius	127
FpM_red	107	FpXQXQ_inv	126
FpM_sub	107	FpXQXQ_invsafe	126
FpM_suppl	109	FpXQXQ_matrix_pow	127
FpM_to_mod	144	FpXQXQ_minpoly	127
FpVV_to_mod	144	FpXQXQ_mul	126
FpV_add	107	FpXQXQ_pow	126
FpV_dotproduct	108	FpXQXQ_powers	127
FpV_dotsquare	108	FpXQXQ_sqr	126
FpV_factorback	108	FpXQXV_FpXQX_fromdigits	125
FpV_FpC_mul	108	FpXQXV_prod	125
FpV_FpMs_mul	175	FpXQX_ddf	128
FpV_FpM_polint	117	FpXQX_ddf_degree	128
FpV_inv	97	FpXQX_degfact	128
FpV_invVandermonde	117	FpXQX_digits	125
FpV_polint	117	FpXQX_div	125
FpV_red	107	FpXQX_divrem	125
FpV_roots_to_pol	116	FpXQX_div_by_X_x	125
FpV_sub	107	FpXQX_dotproduct	125
FpV_to_mod	144	FpXQX_extgcd	125
FpXC_center	122	FpXQX_factor	128
FpXC_FpXQV_eval	122	FpXQX_factor_squarefree	128
FpXM_center	122	FpXQX_FpXQXQV_eval	126

FpXQX_FpXQXQ_eval	125	FpXQ_red	119
FpXQX_FpXQ_mul	125	FpXQ_sqr	120
FpXQX_Frobenius	129	FpXQ_sqrt	120
FpXQX_gcd	125	FpXQ_sqrtn	120, 121, 134
FpXQX_get_red	126	FpXQ_sub	119
FpXQX_halfgcd	125	FpXQ_trace	121
FpXQX_invBarrett	125	FpXT_red	114
FpXQX_isplayer	128	FpXV_FpC_mul	116
FpXQX_mul	125	FpXV_FpX_fromdigits	115
FpXQX_nbfact	129	FpXV_prod	116
FpXQX_nbfact_Frobenius	129	FpXV_red	114
FpXQX_nbroots	129	FpXX_add	122
FpXQX_normalize	124	FpXX_deriv	123
FpXQX_powu	125	FpXX_FpX_mul	122
FpXQX_red	125	FpXX_Fp_mul	122
FpXQX_rem	125	FpXX_halve	123
FpXQX_renormalize	125	FpXX_integ	123
FpXQX_roots	128	FpXX_mulu	122
FpXQX_split_part	128	FpXX_neg	122
FpXQX_sqr	125	FpXX_red	122
FpXQX_to_mod	125	FpXX_renormalize	122
FpXQ_add	119	FpXX_sub	122
FpXQ_autpow	121	FpXYQQ_pow	123
FpXQ_outpowers	122	FpXY_eval	123
FpXQ_outsum	121	FpXY_evalx	123
FpXQ_outtrace	121	FpXY_evaly	123
FpXQ_charpoly	121	FpXY_FpXQV_evalx	123
FpXQ_conjvec	121	FpXY_FpXQ_evalx	123
FpXQ_div	120	FpXY_Fq_evaly	123
FpXQ_ellcard	323	FpX_add	114
FpXQ_elldivpol	323	FpX_center	115, 116
FpXQ_ellgens	323	FpX_center_i	116
FpXQ_ellgroup	323	FpX_chinese_coprime	117
FpXQ_ellj	323	FpX_convolution	115
FpXQ_elljissupersingular	323	FpX_ddf	117
FpXQ_elltwist	323	FpX_ddf_degree	117
FpXQ_ffisom_inv	129	FpX_degfact	117, 131, 140
FpXQ_inv	120	FpX_deriv	115
FpXQ_invsafe	120	FpX_digits	115
FpXQ_issquare	120	FpX_disc	118
FpXQ_log	120, 121, 134	FpX_div	115
FpXQ_matrix_pow	121	FpX_divrem	115
FpXQ_minpoly	121	FpX_div_by_X_x	115
FpXQ_mul	120	FpX_dotproduct	116
FpXQ_norm	121	FpX_eval	116
FpXQ_order	120, 134	FpX_extgcd	115
FpXQ_pow	120	FpX_factor	117
FpXQ_powers	121	FpX_factorff	128
FpXQ_powu	120	FpX_factorff_irred	128, 131

FpX_factor_squarefree	117	FpX_resultant	118
FpX_ffintersect	129	FpX_roots	118
FpX_ffisom	128, 131	FpX_rootsff	128, 131
FpX_FpC_nfpoleval	274	FpX_split_part	118
FpX_FpV_multieval	116	FpX_sqr	115
FpX_FpXQV_eval	122	FpX_sub	115
FpX_FpXQ_eval	122	FpX_to_mod	144
FpX_FpXY_resultant	118	FpX_translate	115
FpX_Fp_add	116	FpX_valrem	115
FpX_Fp_add_shallow	116	Fp_2gener	98
FpX_Fp_mul	116	Fp_add	15, 97
FpX_Fp_mulspec	116	Fp_addmul	97
FpX_Fp_mul_to_monich	116	Fp_center	97
FpX_Fp_sub	116	Fp_center_i	97
FpX_Fp_sub_shallow	116	Fp_div	97
FpX_Frobenius	116	Fp_ellcard	318
FpX_fromNewton	118	Fp_ellcard_SEA	318
FpX_gcd	115	Fp_elldivpol	318
FpX_gcd_check	286	Fp_ellgens	318
FpX_get_red	118	Fp_ellgroup	318
FpX_halfgcd	115	Fp_ellj	317
FpX_halve	115	Fp_elljissupersingular	317
FpX_integ	115	Fp_elltwist	318
FpX_invBarret	118	Fp_factored_order	98
FpX_invBarrett	116	Fp_ffellcard	318
FpX_invLaplace	118	Fp_FpXQ_log	120
FpX_isplayer	117	Fp_FpX_sub	116
FpX_is_irred	117, 140	Fp_halve	97
FpX_is_squarefree	117	Fp_inv	97
FpX_is_totally_split	117	Fp_invgen	97
FpX_Laplace	118	Fp_invsafe	97
FpX_matFrobenius	116	Fp_isplayer	98
FpX_mul	115	Fp_issquare	98
FpX_mulspec	115	Fp_log	98, 121
FpX_mulu	116	Fp_modinv_to_j	324
FpX_nbfact	117	Fp_mul	97
FpX_nbfact_Frobenius	117	Fp_muls	97
FpX_nbroots	117	Fp_mulu	97
FpX_neg	114	Fp_neg	97
FpX_Newton	118	Fp_order	98
FpX_normalize	116	Fp_polmodular_evalx	324
FpX_oneroot	118	Fp_pow	97
FpX_oneroot_split	118	Fp_powers	97
FpX_powu	115	Fp_pows	97
FpX_ratlift	149	Fp_powu	97
FpX_red	114	Fp_pow_init	97
FpX_rem	115	Fp_pow_table	97
FpX_renormalize	115	Fp_ratlift	149
FpX_rescale	116	Fp_red	96

Fp_sqr	97	FqX_ddf	128
Fp_sqrt	98	FqX_degfact	128
Fp_sqrtn	98	FqX_deriv	124
Fp_sqrt_i	98	FqX_div	124
Fp_sub	97	FqX_divrem	124
Fp_to_mod	144	FqX_div_by_X_x	124
FqC_add	109	FqX_eval	124
FqC_Fq_mul	109	FqX_extgcd	124
FqC_sub	109	FqX_factor	128
FqC_to_FlxC	154	FqX_factor_squarefree	128
FqM_deplin	109	FqX_Fp_mul	124
FqM_det	109	FqX_Fq_add	123
FqM_FqC_gauss	109	FqX_Fq_mul	124
FqM_FqC_invimage	109	FqX_Fq_mul_to_monic	124
FqM_FqC_mul	109	FqX_Fq_sub	123
FqM_gauss	109	FqX_gcd	124
FqM_image	109	FqX_get_red	126
FqM_indexrank	109	FqX_halfgcd	124
FqM_inv	109	FqX_halve	124
FqM_invimage	109	FqX_integ	124
FqM_ker	109	FqX_ispower	128
FqM_mul	109	FqX_is_squarefree	128
FqM_rank	109	FqX_mul	124
FqM_suppl	109	FqX_mulu	124
FqM_to_FlxM	155	FqX_nbfact	129
FqM_to_mod	144	FqX_nbroots	129
FqM_to_nfM	283	FqX_neg	123
FqV_inv	121	FqX_normalize	124
FqV_red	119	FqX_powu	124
FqV_roots_to_pol	128	FqX_red	119
FqV_to_FlxV	155	FqX_rem	124
FqV_to_nfV	283	FqX_roots	128
FqXn_exp	126	FqX_sqr	124
FqXn_inv	126	FqX_sub	123
FqXn_mul	126	FqX_to_FFX	227
FqXn_sqr	126	FqX_to_mod	125
FqXQ_add	127	FqX_to_nfX	283
FqXQ_div	127	FqX_translate	124
FqXQ_inv	127	Fq_add	120
FqXQ_invsafe	127	Fq_div	120
FqXQ_matrix_pow	128	Fq_ellcard_SEA	323
FqXQ_mul	127	Fq_elldivpolmod	323
FqXQ_pow	127	Fq_Fp_mul	120
FqXQ_powers	127	Fq_halve	120
FqXQ_sqr	127	Fq_inv	120
FqXQ_sub	127	Fq_invsafe	120
FqXY_eval	124	Fq_ispower	121
FqXY_evalx	124	Fq_issquare	121
FqX_add	123	Fq_log	121

Fq_mul	120	gal_get_mod	295
Fq_mulu	120	gal_get_orders	295
Fq_neg	120	gal_get_p	295
Fq_neg_inv	120	gal_get_pol	295
Fq_pow	121	gal_get_roots	295
Fq_powu	121	gammamellininv	328
Fq_red	119	gammamellininvinit	328
Fq_sqr	120	gammamellininvrt	328
Fq_sqrt	121	gand	212
Fq_sqrtn	121	garbage collecting	17
Fq_sub	120	gassoc_proto	99
Fq_to_FF	227	gaussred_from_QR	173
Fq_to_FpXQ	119	gbezout	216
Fq_to_nf	283	gboundcf	96
fractor	199	gcdii	96
Frobeniusform	173	gceil	209
fromdigitsu	86	gclone	27, 67, 68
fromdigits_2k	86	gcloneref	68
fujiwara_bound	225	gclone_refc	69
fujiwara_bound_real	225	gcmp	210
fun(E, ell)	42	gcmpgs	211
fun(E, H)	42	gcmpsg	211
functions_basic	52	gcoeff	15, 61, 252
functions_default	52	gconj	223
functions_gp	52	gcopy	27, 68
fuse_Z_factor	161	gcopy_avma	67
f_PRETTYMAT	239	gcopy_lg	68
f_RAW	239, 240	gcvtoi	209
f_TEX	239, 240	gcvtop	200
G			
gabs[z]	217	gc_needed	23
gadd	81, 218	gdeuc	215
gaddgs	15, 218	gdiv	218
gaddsg	15, 218	gdiventgs[z]	215
gaddz	15, 26, 81, 219	gdiventres	215
gadd[z]	81	gdiventsg	215
gaffect	26, 27, 199	gdivent[z]	215
gaffsg	27, 199	gdivexact	214
galoisexport	237	gdivgs	218
galoisidentify	237	gdivmod	215
galoisinit	235, 294, 295	gdivround	215
galois_group	235	gdivsg	218
gal_get_den	295	gdivz	219
gal_get_e	295	gdvd	215
gal_get_gen	295	gel	14, 15, 61, 252
gal_get_group	295	GEN	13
gal_get_invvdm	295	GENbinbase	65
		gener_F2xq	141
		gener_Flxq	135
		gener_FpXQ	121

gener_FpXQ_local	121	gen_pow_table	219
gener_Fq_local	121	gen_product	219
GENtoGENstr	239	gen_RgX_bkeval	195
GENtoGENstr_nospace	239	gen_search	214
GENtostr	39, 239	gen_select_order	192
GENtostr_raw	239	gen_setminus	213
GENtostr_unquoted	239	gen_Shanks	192
GENtoTeXstr	39, 239	gen_Shanks_init	192
gen_0	13, 32	gen_Shanks_log	191
gen_1	13	gen_Shanks_sqrtn	192
gen_2	13	gen_sort	213
gen_bkeval	195	gen_sort_inplace	213
gen_bkeval_powers	195	gen_sort_uniq	213
gen_cmp_RgX	214	gen_ZpM_Dixon	176
gen_det	194	gen_ZpX_Dixon	197
gen_digits	196	gen_ZpX_Newton	197
gen_ellgens	193, 315	geq	212
gen_ellgroup	193	gequal	183, 210
gen_factorback	220	gequal0	211
gen_factored_order	192	gequal1	211
gen_FpM_Wiedemann	176	gequalgs	211
gen_fromdigits	196	gequalm1	211
gen_Gauss	194	gequalsg	211
gen_Gauss_pivot	194	gequalX	210
gen_gener	192	gerepile	18, 20, 26, 27, 65, 91
gen_indexsort	213	gerepileall	23
gen_indexsort_uniq	213	gerepileall	20, 23, 66
gen_ker	194	gerepileallsp	20, 66
gen_m1	13	gerepilecoeffs	66
gen_m2	13	gerepilecoeffssp	66
gen_matcolinvimage	194	gerepilecopy	20, 23, 66
gen_matcolmul	194	gerepilemany	66
gen_matid	194	gerepilemanysp	66
gen_matinvimage	194	gerepileupto	19, 20, 25, 27, 66, 91, 158, 205, 207, 252, 279
gen_matmul	194	gerepileuptoint	66
gen_order	192	gerepileuptoleaf	66
gen_PH_log	192	getheap	69
gen_plog	192	getrand	96
gen_Pollard_log	192	getrealprecision	231
gen_pow	219	gettime	41
gen_powers	195, 220	get_arith_Z	192
gen_powu	219	get_arith_ZZM	192
gen_powu_fold	219	get_bnf	267
gen_powu_fold_i	219	get_bnfpol	267
gen_powu_i	219	get_F2xqE_group	193
gen_pow_fold	219	get_F2xqX_degree	143
gen_pow_fold_i	219	get_F2xqX_mod	143
gen_pow_i	219	get_F2xqX_var	143
gen_pow_init	219		

get_F2xq_field	194	ginvmod	215
get_F2x_degree	139	gisexactzero	211
get_F2x_mod	138	glcm	216
get_F2x_var	139	gle	212
get_FlxqE_group	193	glt	212
get_FlxqXQ_algebra	196	gmael	15, 61
get_FlxqX_degree	136	gmael1	15
get_FlxqX_mod	136	gmael2	61
get_FlxqX_var	136	gmael3	61
get_Flxq_field	194	gmael4	61
get_Flxq_star	193	gmael5	61
get_Flx_degree	129	gmax	210
get_Flx_mod	129	gmaxgs	211
get_Flx_var	129	gmaxsg	211
get_Fl_red	79	gmax_shallow	210
get_FpE_group	193	gmin	210
get_FpXQE_group	193	gmings	211
get_FpXQXQ_algebra	196	gminsg	211
get_FpXQX_algebra	196	gmin_shallow	210
get_FpXQX_degree	126	gmodgs	215
get_FpXQX_mod	126	gmodsg	215
get_FpXQX_var	126	gmodulgs	201
get_FpXQ_algebra	196	gmodulo	200
get_FpXQ_star	193	gmodulsg	201
get_FpX_algebra	196	gmodulss	201
get_FpX_degree	119	gmod[z]	215
get_FpX_mod	119	gmul	218
get_FpX_var	119	gmul2n[z]	209
get_Fp_field	194	gmulgs	218
get_Fq_field	194	gmulsg	218
get_lex	260	gmulz	219
get_modpr	268	gne	212
get_nf	267	gneg[z]	217
get_nfpol	267	gneg_i	217
get_nf_field	194	gnorml1	220
get_prid	268	gnorml1_fake	220
get_Rg_algebra	196	gnorml2	220
gexpo	30, 58	gnot	212
gexpo_safe	58	gor	212
gfloor	209	<i>GP prototype</i>	71
gfrac	209	gphelp_keyword_list	55
ggamma1m1	232	gpininstall	54
ggcd	216	gpow	218
gge	212	gpowers	218
ggt	212	gpowgs	218
ghalf	13	gprec	200
gidentical	174, 210	gprecision	59
gimag	223	gprec_w	200
ginv	217	gprec_wensure	200

gprec_wtrunc	200	group_ident	237
gprimepi_lower_bound	162	group_ident_trans	237
gprimepi_upper_bound	162	group_isA4S4	236
gp_alarm_handler	55	group_isabelian	236
gp_call	261	group_leftcoset	236
gp_call2	261	group_order	236
gp_callbool	261	group_perm_normalize	236
gp_callprec	261	group_quotient	236
gp_callvoid	261	group_rightcoset	236
gp_context_restore	55	group_set	235
gp_context_save	55	group_subgroups	236
gp_echo_and_log	55	group_subgroup_isnormal	236
gp_eval	261	group_to_cc	236
gp_evalbool	261	gshift[z]	209
gp_evalprec	261	gsigne	30, 58
gp_evalupto	261	gsincos	232
gp_evalvoid	261	gsizebyte	26
gp_filter	54	gsizeword	26
gp_format_prompt	54	gsmith	302
gp_format_time	54	gsmithall	302
gp_handle_exception	53	gsprintf	240
gp_help	54	gsqr	217
gp_load_gprc	54	gsqrpowers	219
gp_meta	54	GSTR	34
gp_read_file	38, 54	gsub	218
gp_read_str	36, 37, 54, 74	gsubgs	218
gp_read_stream	38	gsubsg	218
gp_read_str_multiline	37	gsubst	221
gp_sigint_fun	53	gsubz	219
Gram matrix	171	gsupnorm	220
gram_matrix	171	gsupnorm_aux	220
greal	223	gtocol	203
gred_rfac2	33	gtodouble	28, 199
grem	215	gtofp	27, 200
grndtoi	209	gtolong	28, 199
grootsof1	218	gtomat	203
ground	209	gtomp	200
groupelts_abelian_group	237	gtopoly	201
groupelts_center	237	gtopolyrev	201
groupelts_conjclasses	235	gtos	199
groupelts_exponent	237	gtoser	202
groupelts_set	235	gtoser_prec	202, 203
group_abelianHNF	236	gtovec	203
group_abelianSNF	236	gtovecsmall	203
group_domain	235	gtrans	253
group_elts	235	gtrunc	209
group_export	237	gtrunc2n	85, 209
group_export_GAP	237	gunclone	27, 69
group_export_MAGMA	237	gunclone_deep	69

<code>gval</code>	210	<code>hnfmod</code>	302
<code>gvaluation</code>	209	<code>hnfmodid</code>	302
<code>gvar</code>	32, 35, 58	<code>hnfperm</code>	302
<code>gvar2</code>	58	<code>hnf_CENTER</code>	300
<code>gvsprintf</code>	240	<code>hnf_divscale</code>	300
<code>G_ZGC_mul</code>	174	<code>hnf_invimage</code>	301
<code>G_ZG_mul</code>	174	<code>hnf_invscale</code>	300
H			
<code>hamming1</code>	77	<code>hnf_MODID</code>	299
<code>hashentry</code>	249	<code>hnf_PART</code>	299
<code>hashtable</code>	249	<code>hnf_solve</code>	301
<code>hash_create</code>	250	<code>hqfeval</code>	221
<code>hash_create_str</code>	250	<code>hyperell_locally_soluble</code>	324
<code>hash_create_ulong</code>	250	<code>h_APROPOS</code>	55
<code>hash_dbg</code>	251	<code>h_LONG</code>	54
<code>hash_destroy</code>	251	<code>h_REGULAR</code>	54
<code>hash_GEN</code>	251	I	
<code>hash_haskey_long</code>	250	<code>icopy</code>	83
<code>hash_init_GEN</code>	250	<code>icopyifstack</code>	68
<code>hash_insert</code>	250	<code>icopyspec</code>	83
<code>hash_insert2</code>	250	<code>icopy_avma</code>	67
<code>hash_insert_long</code>	250	<code>idealadd</code>	279
<code>hash_keys</code>	251	<code>idealaddmultoone</code>	280
<code>hash_remove</code>	250	<code>idealaddtoone</code>	280
<code>hash_remove_select</code>	250	<code>idealaddtoone_i</code>	280
<code>hash_search</code>	250	<code>idealaddtoone_raw</code>	280
<code>hash_search2</code>	250	<code>idealappr</code>	280, 294
<code>hash_select</code>	250	<code>idealappr0</code>	294
<code>hash_str</code>	250, 251	<code>idealapprfact</code>	280
<code>hash_str2</code>	251	<code>idealchinese</code>	280
<code>hash_values</code>	251	<code>idealchineseinit</code>	280
<code>hclassno</code>	296	<code>idealcoprime</code>	280
<code>hclassno6</code>	296	<code>idealcoprimefact</code>	280
<code>hclassno6u</code>	296	<code>idealdiv</code>	279
<code>heap</code>	14	<code>idealdivexact</code>	279
<code>hexadecimal tree</code>	40	<code>idealdivpowprime</code>	279
<code>HIGHBIT</code>	62	<code>idealfactor</code>	278, 280
<code>HIGHEXPBIT</code>	62	<code>idealfrobenius_aut</code>	282
<code>HIGHMASK</code>	62	<code>idealhnf</code>	278
<code>HIGHVALPBIT</code>	62	<code>idealhnf0</code>	278
<code>HIGHWORD</code>	62	<code>idealHNF_inv</code>	280
<code>hilbertii</code>	100	<code>idealHNF_inv_Z</code>	280
<code>hnf</code>	302	<code>idealHNF_mul</code>	279
<code>hnfall</code>	302	<code>idealhnf_principal</code>	278
<code>hnfdivide</code>	301	<code>idealhnf_shallow</code>	278
<code>hnflll</code>	302	<code>idealhnf_two</code>	279
<code>hnfmerge_get_1</code>	280	<code>idealHNF_Z_factor</code>	281
		<code>idealHNF_Z_factor_i</code>	281

idealinv	279	indices_to_vec01	284
ideallog	278	infinity	34
idealmoddivisor	290	inf_get_sign	34
idealmul	279	initprimes	64
idealmulpowprime	279	initprimetable	64
idealmulred	279, 287	init_Fq	128
idealpow	279	init_primepointer_geq	64
idealpowred	279	init_primepointer_gt	64
idealpows	279	init_primepointer_leq	64
idealprimedec	278, 280, 281	init_primepointer_lt	64
idealprimedec_degrees	281	input	37
idealprimedec_galois	281	install	36, 40, 73, 75
idealprimedec_kummer	281	int2n	82
idealprimedec_limit_f	281	int2u	82
idealprimedec_limit_norm	281	int2um1	82
idealprincipalunits	285	integer	29
idealprod	279	integser	226
idealprodprime	279	int_LSW	29
idealprodval	279	int_MSW	29
idealpseudomin	289	int_nextW	29
idealpseudomin_nonscalar	289	int_normalize	29
idealpseudored	289	int_precW	29
idealramfrobenius	282	int_W	29
idealramfrobenius_aut	282	int_W_lg	29
idealramgroups_aut	282	invmod	98
idealred	289	invmod2BIL	78
idealred0	289	invr	91
idealred_elt	289	inv_content	217
idealsqr	279	isclone	28
idealstar	285	iscomplex	212
Idealstarprk	290	isexactzero	210
idealtyp	268	isinexact	211
identity_perm	234	isinexactreal	211
id_MAT	268	isint	212
id_PRIME	268	isint1	211
id_PRINCIPAL	268	isintm1	211
ifac_isprime	162	isintzero	211
ifac_next	161	ismpzero	211
ifac_read	161	isonstack	68
ifac_skip	161	isprime	163
ifac_start	161	isprimeAPRCL	163
image	176	isprimeECP	163
image2	176	isprimepower	100
imag_i	223	isprincipal	287
indefinite binary quadratic form	33	isprincipalfact	287
indexlexsort	213	isprincipalfact_or_fail	287
indexpartial	286	isprincipalforce	294
indexsort	213	isprincipalgen	294
indexvecsort	213	isprincipalgenforce	294

mfmatembed	333	mkcol4s	205
mfnewdim	333	mkcol5	206
MFnew_get_vj	332	mkcol6	206
mfnucusp	331	mkcolcopy	204
mfnucusp_fact	331	mkcoln	25, 208
mfnucusp_fact	331	mkcols	205
mfolddim	333	mkcomplex	205
mfsturmNgk	333	mkerr	207
mfsturmNk	333	mkfrac	206
mfsturm_mf	333	mkfraccopy	204
mftobasisES	333	mkfracsss	204
mftocol	334	mkintmod	206
mfvecembed	333	mkintmodu	204
mfvectomat	334	mkintn	25, 26, 84, 208
MF_get_basis	331, 332	mkmat	206
MF_get_CHI	331	mkmat2	206
mf_get_CHI	332	mkmat22	206
MF_get_dim	331	mkmat22s	205
MF_get_E	331	mkmat3	206
mf_get_field	332	mkmat4	206
MF_get_fields	332	mkmat5	206
MF_get_gk	331	mkmatcopy	204
mf_get_gk	332	mkmoo	34
MF_get_gN	331	mkoo	34
mf_get_gN	332	mkpolmod	206
MF_get_k	331	mkpoln	25, 208
mf_get_k	332	mkqfi	207
MF_get_M	332	mkquad	205
MF_get_Mindex	332	mkfrac	206
MF_get_Minv	332	mkfraccopy	204
MF_get_N	331	mkvec	206
mf_get_N	332	mkvec2	206
MF_get_newforms	332	mkvec2copy	204
mf_get_NK	332	mkvec2s	205
MF_get_r	331	mkvec3	206
mf_get_r	332	mkvec3s	205
MF_get_S	331, 332	mkvec4	207
MF_get_space	331	mkvec4s	205
mf_get_type	333	mkvec5	207
millerrabin	163	mkvecscopy	204
mindd	88	mkvecn	25, 208
minss	88	mkvecs	205
minuu	88	mkvecsmall	205
mkcol	206	mkvecsmall2	205
mkcol2	206	mkvecsmall3	205
mkcol2s	205	mkvecsmall4	205
mkcol3	206	mkvecsmalln	205
mkcol3s	205	Mod16	94
mkcol4	206	mod16	94

<code>nfalgtobasis</code>	275	<code>nfsqr</code>	274
<code>nfarchstar</code>	285	<code>nfsqri</code>	275
<code>nfbasistoalg</code>	275	<code>nfsub</code>	274
<code>nfchecksigns</code>	284	<code>nftrace</code>	274
<code>nfcyclotomicunits</code>	293	<code>nftyp</code>	267
<code>nfC_multable_mul</code>	275	<code>nfval</code>	274, 284
<code>nfC_nf_mul</code>	275	<code>nfV_to_FqV</code>	283
<code>nfdiv</code>	274	<code>nfX_to_FqX</code>	283
<code>nfdiveuc</code>	274	<code>nf_deg1_prime</code>	282
<code>nfdivrem</code>	274	<code>nf_FORCE</code>	287
<code>nfeltup</code>	292	<code>nf_GEN</code>	287
<code>nfembed</code>	284	<code>nf_GENMAT</code>	287
<code>nffactorback</code>	278	<code>nf_GEN_IF_PRINCIPAL</code>	287
<code>nfgaloisconj</code>	293	<code>nf_get_allroots</code>	269
<code>nfgaloismatrix</code>	295	<code>nf_get_degree</code>	269
<code>nfgaloispermtobasis</code>	295	<code>nf_get_diff</code>	269
<code>nfgcd</code>	182	<code>nf_get_disc</code>	269
<code>nfgcd_all</code>	182	<code>nf_get_G</code>	269
<code>nfgwkummer</code>	292	<code>nf_get_Gtwist</code>	288, 289, 304
<code>nfinit_basic</code>	286	<code>nf_get_Gtwist1</code>	289
<code>nfinit_complete</code>	286	<code>nf_get_index</code>	269
<code>nfinv</code>	274	<code>nf_get_invzk</code>	269
<code>nfinvmodideal</code>	275	<code>nf_get_M</code>	269, 288
<code>nfissquarefree</code>	292	<code>nf_get_pol</code>	268
<code>nfmaxord</code>	285	<code>nf_get_prec</code>	269
<code>nfmaxord_t</code>	285, 286	<code>nf_get_r1</code>	269
<code>nfmaxord_to_nf</code>	286	<code>nf_get_r2</code>	269
<code>nfmod</code>	275	<code>nf_get_ramified_primes</code>	269
<code>nfmodprinit</code>	282, 283	<code>nf_get_roots</code>	269
<code>nfmul</code>	274	<code>nf_get_roundG</code>	269, 289
<code>nfmuli</code>	275	<code>nf_get_sign</code>	269
<code>nfM_det</code>	276	<code>nf_get_Tr</code>	269
<code>nfM_inv</code>	276	<code>nf_get_varn</code>	268
<code>nfM_mul</code>	276	<code>nf_get_zk</code>	269
<code>nfM_nfC_mul</code>	276	<code>nf_get_zkden</code>	269
<code>nfM_to_FqM</code>	283	<code>nf_get_zkprimpart</code>	269
<code>nfnewprec</code>	273	<code>nf_hyperell_locally_soluble</code>	324
<code>nfnewprec_shallow</code>	273	<code>nf_nfzk</code>	292
<code>nfnorm</code>	274	<code>nf_PARTIALFACT</code>	286
<code>nfpoleval</code>	274	<code>nf_pV_to_prV</code>	282
<code>nfpow</code>	274	<code>nf_rnfeq</code>	292
<code>nfpowmodideal</code>	275	<code>nf_rnfeqsimple</code>	292
<code>nfpow_u</code>	274	<code>nf_ROUND2</code>	286
<code>nfrootsof1</code>	293	<code>nf_to_Fp_coprime</code>	290
<code>nfroots_if_split</code>	293	<code>nf_to_Fq</code>	283
<code>nfsign</code>	278, 284	<code>nf_to_Fq_init</code>	282
<code>nfsign_arch</code>	278, 284	<code>nf_to_scalar_or_alg</code>	275
<code>nfsign_from_logarch</code>	285	<code>nf_to_scalar_or_basis</code>	275
<code>nfsign_units</code>	285	<code>nmV_chinese_center</code>	148

nmV_chinese_center_tree	148	padicprec	152		
nm_Z_mul	156	padicprec_relative	152		
normalize	203	padic_to_Fl	154		
normalizapol	33, 201	padic_to_Fp	106		
normalizapol_approx	201	padic_to_Q	152		
normalizapol_lg	201	padic_to_Q_shallow	152		
normalize_frac	60	paricfg_buildinfo	76		
NO_VARIABLE	32, 35, 58, 62	paricfg_compiledat	76		
numberofconjugates	293	paricfg_datadir	76		
numdivu	100	paricfg_gphelp	76		
numdivu_fact	100	paricfg_mt_engine	76		
numerr_name	249	paricfg_vcsversion	76		
numer_i	216	paricfg_version	76		
nv_fromdigits_2k	86	paricfg_version_code	76		
nxCV_chinese_center	148	pariErr	241		
nxMV_chinese_center	148	PariOUT	240		
nxV_chinese_center	148	pariOut	241		
nxV_chinese_center_tree	148	paristack_newrsize	52		
		paristack_resize	51		
O				paristack_setsize	51
obj	265	parivstack_reset	51		
obj_check	265	parivstack_resize	52		
obj_checkbuild	265	pari_add_defaults_module	52		
obj_checkbuild_padicprec	265	pari_add_function	52		
obj_checkbuild_prec	265	pari_add_hist	55		
obj_checkbuild_realprec	265	pari_add_module	52		
obj_free	266, 313	pari_alarm	54		
obj_init	265	pari_ask_confirm	54		
obj_insert	265	pari_calloc	17		
obj_insert_shallow	265	pari_CATCH	44		
obj_reinit	265	pari_CATCH_reset	45		
odd	78	pari_center	54		
odd_prime_divisors	159	pari_close	49		
omega	160	pari_close_opts	51		
omegau	100	pari_community	54		
ONLY_DIVIDES	104, 185	pari_compile_str	54		
ONLY_REM	104, 185	pari_daemon	51		
outmat	38	pari_ENDCATCH	45		
output	38	pari_err	34, 39, 45, 244, 263		
output	38, 40, 242	pari_err2str	249		
out_printf	241	pari_errfile	241		
out_putc	241	pari_err_last	45		
out_puts	241	pari_err_TYPE	312		
out_term_color	242	pari_fclose	243		
out_vprintf	241	pari_flush	38, 241		
		pari_fopen	243		
P				pari_fopengz	243
p-adic number	31	pari_fopen_or_fail	243		
		pari_fprintf	39		

pari_fread_chars	242	pari_strdup	239
pari_free	17, 65	pari_strndup	239
pari_get_hist	55	pari_thread_alloc	339
pari_get_histtime	55	pari_thread_close	340
pari_get_homedir	243	pari_thread_free	339
pari_hit_return	54	pari_thread_init	339
pari_infile	54	pari_thread_start	339
pari_init	13, 14, 49	pari_thread_sync	339
pari_init_opts	49	pari_thread_valloc	339
pari_init_primes	50, 51	pari_timer	41
pari_is_default	261	pari_TRY	45
pari_is_dir	242	pari_unique_dir	244
pari_is_file	242	pari_unique_filename	244
pari_kernel_close	50	pari_unique_filename_suffix	244
pari_kernel_init	50	pari_unlink	242
pari_kill_plot_engine	335	pari_var_close	70
pari_last_was_newline	241	pari_var_create	70
pari_library_path	54	pari_var_init	70
pari_malloc	17, 65, 247	pari_var_next	70
pari_mt_close	51	pari_var_next_temp	70
pari_mt_init	50	PARI_VERSION	76
pari_nb_hist	55	pari_version	76
PARI_OLD_NAMES	14	PARI_VERSION_SHIFT	76
pari_outfile	38, 241	pari_vfprintf	39
PARI_plot	335	pari_vprintf	39
pari_plot_by_file	336	pari_vsprintf	39
pari_printf	38, 39, 40, 71, 241, 242	pari_warn	40
pari_print_version	54	parser code	74
pari_putc	38, 71, 241	path_expand	243
pari_puts	38, 71, 241, 242	perm_commute	234
pari_rand	96	perm_conj	234
pari_realloc	17, 247	perm_cycles	234
pari_RETRY	45	perm_inv	234
pari_safefopen	243	perm_mul	234
pari_set_last_newline	241	perm_order	234
pari_set_plot_engine	335	perm_pow	234
pari_sighandler	51	perm_sign	234
pari_sig_init	51	perm_to_Z	234
pari_sp	17	pgener_Fl	79
pari_sprintf	39, 239	pgener_Fl_local	79
pari_stackcheck_init	51	pgener_Fp	99
pari_stack_alloc	252	pgener_Fp_local	99
pari_stack_base	252	pgener_Zl	79
pari_stack_delete	252	pgener_Zp	99
pari_stack_init	252	Pi2n	233
pari_stack_new	252	PiI2	233
pari_stack_pushp	252	PiI2n	234
pari_stdin_isatty	243	plotbox	335
pari_str	240	plotclip	335

plotcolor	335	polredabsall	294
plotcopy	335	Polrev	203
plotcursor	335	polx_F2x	139
plotdraw	335	polx_F2xX	142
plotth	335	polx_Flx	131
plotthraw	335	polx_FlxX	135
plotsizes	335	polx_zx	182
plotinit	335	polynomial	32
plotkill	335	pol_0	203
plotline	335	pol_1	203
plotlines	335	pol_x	203
plotlinetype	335	pol_xn	203
plotmove	336	pol_xnall	203
plotpoints	336	pol_x_powers	203
plotpointsize	336	pop_lex	73, 260
plotpointtype	336	power series	33
plotrbox	336	powersr	95
plotrecth	335	powgi	218, 297
plotrecthraw	336	powii	94
plotrline	336	powis	94
plotrmove	336	powIs	95
plotrpoint	336	powiu	94
plotscale	336	powfrac	95
plotstring	336	powrs	94
point_to_a4a6	313	powrshalf	95
point_to_a4a6_F1	313	powru	95
pol0_F2x	139	powruhalf	95
pol0_Flx	131	powuu	94
pol1_F2x	139	ppg	160
pol1_F2xX	142	ppi	160
pol1_Flx	131	pple	160
pol1_FlxX	135	ppo	160
polclass	324	prec2nbits	57
polcoef_i	224	prec2nbits_mul	57
poldivrem	215	prec2ndec	56
poleval	185, 221	precdbl	57
polgalois	237	precision	59
polhensellift	150, 151	precpr	31, 58
polmod	32	PRECPBITS	62
polmodular	324	PRECPSHIFT	62
polmodular_ZM	324	preferences file	74
polmodular_ZXX	324	<i>prid</i>	280
polmod_nffix	293	prime	162
polmod_nffix2	293	primeform	298
polmod_to_embed	225	primeform_u	298
Polred	294	primepi_lower_bound	162
polred0	294	primepi_upper_bound	162
polredabs	294	primes	162
polredabs2	294	primes_interval	162

primes_interval_zv	162	qfevalb	221
primes_upto_zv	162	qfi	296
primes_zv	162	qficomp	296
prime_fact	204	qficompraw	296
primitive_root	79	qfipowraw	297
primitive_part	216	qfisolvep	297
primpart	216	qfisqr	296
printf	38, 39, 71	qfisqrraw	297
print_fun_list	55	qfi_1	296
prV_lcm_capZ	282	qfi_log	297
pr_basis_perm	283	qfi_order	297
pr_equal	284	qfi_Shanks	297
pr_get_e	281	qflll0	303
pr_get_f	281	qflllgram0	303
pr_get_gen	281	qfr	296
pr_get_p	281	qfr3	298
pr_get_tau	281	qfr3_comp	298
pr_hnf	281	qfr3_pow	298
pr_inv	281	qfr3_red	298
pr_inv_p	281	qfr3_rho	298
pr_is_inert	281	qfr3_to_qfr	299
pr_norm	281	qfr5	298
pr_uniformizer	282	qfr5_comp	299
psdraw	336	qfr5_dist	299
psploth	336	qfr5_pow	299
psplothraw	336	qfr5_red	299
pthread_join	339	qfr5_rho	299
push_lex	73, 260	qfr5_to_qfr	299
putch	240	qfrcomp	296
puts	240	qfrcompraw	297
p_to_FF	226	qfrpow	297
p_to_FF(p,0)	227	qfrpowraw	297

Q

Qdivii	216	qfrsqr	296
Qevproj_apply	168	qfrsqrraw	297
Qevproj_apply_vecei	168	qfr_1	296
Qevproj_down	168	qfr_data_init	298
Qevproj_init	168	qfr_to_qfr5	299
qfbcompraw	297	qf_apply_RgM	221
qfbforms	224	qf_apply_ZM	221
qfbpowraw	297	QM_charpoly_ZX	168
qfbsolve	297	QM_charpoly_ZX_bound	168
qfb_apply_ZM	224	QM_det	169
qfb_disc	224	QM_gauss	168
qfb_disc3	224	QM_ImQ_hnf	302
qfb_equal1	296	QM_ImQ_hnfall	302
qfeval	221	QM_ImZ_hnf	302
		QM_ImZ_hnfall	302
		QM_indexrank	168

raw	239	retmkqfi	208
rcopy	83	retmkqfr	208
rdivii	92	retmkquad	208
rdiviiz	92	retmkrfrac	208
rdivis	92	retmkvec	207
rdivsi	92	retmkvec2	207
rdivss	92	retmkvec3	207
read	37, 38	retmkvec4	207
readseq	37	retmkvec5	207
real number	30	rfracrecip_to_ser_absolute	202
real2n	82	rfrac_to_ser	202
realprec	57	RgC_add	170
real_0	82	RgC_fpnorml2	172
real_0_bit	82	RgC_gtofp	172
real_1	81	RgC_gtomp	172
real_1_bit	81	RgC_is_ei	172
real_i	223	RgC_is_FFC	227
real_m1	82	RgC_neg	170
real_m2n	82	RgC_RgM_mul	171
rect2ps	336	RgC_RgV_mul	171
rect2ps_i	336	RgC_Rg_add	170
rect2svg	336	RgC_Rg_div	170
reducemodinvertible	305	RgC_Rg_mul	170
reducemodlll	305	RgC_Rg_sub	170
remi2n	91, 177	RgC_sub	170
remlll_pre	79	RgC_to_FpC	106
remll_pre	79	RgC_to_FqC	109
remsBIL	62	RgC_to_nfC	275
residual_characteristic	225	RgE_to_F2xqE	321
<i>resultant (reduced)</i>	150	RgE_to_FlxqE	322
resultant	215, 225	RgE_to_FpE	319
resultant2	225	RgE_to_FpXQE	324
retconst_col	207	RgMrow_RgC_mul	171
retconst_vec	207	RgMrow_zc_mul	156
retmkcol	207	RgMs_structelim	175
retmkcol2	207	RgM_add	170
retmkcol3	207	RgM_Babai	174
retmkcol4	207	RgM_check_ZM	166
retmkcol5	207	RgM_det_triangular	173
retmkcol6	207	RgM_diagonal	172
retmkcomplex	208	RgM_diagonal_shallow	172
retmkfrac	208	RgM_dimensions	170
retmkintmod	208	RgM_fpnorml2	172, 220
retmkmat	207	RgM_Fp_init	106
retmkmat2	207	RgM_gram_schmidt	174
retmkmat3	207	RgM_gtofp	172
retmkmat4	207	RgM_gtomp	172, 173
retmkmat5	208	RgM_Hadamard	173
retmkpolmod	208	RgM_hnfall	302

RgM_inv	173	RgV_check_ZV	164
RgM_invimage	173	RgV_dotproduct	171
RgM_inv_upper	173	RgV_dotsquare	171
RgM_isdiagonal	172	RgV_gtofp	172
RgM_isidentity	172	RgV_isin	172
RgM_isscalar	172	RgV_isscalar	172
RgM_is_FFM	227	RgV_is_FpV	106
RgM_is_FpM	106	RgV_is_QV	164
RgM_is_QM	172	RgV_is_ZMV	169
RgM_is_ZM	172	RgV_is_ZV	164
RgM_minor	253	RgV_is_ZVnon0	164
RgM_mul	171	RgV_is_ZVpos	164
RgM_mulreal	171	RgV_kill0	171
RgM_multosym	171	RgV_neg	170
RgM_neg	170	RgV_nffix	293
RgM_powers	171	RgV_polint	171
RgM_QR_init	173	RgV_prod	171
RgM_rescale_to_int	166	RgV_RgC_mul	170
RgM_RgC_invimage	173	RgV_RgM_mul	171
RgM_RgC_mul	171	RgV_Rg_mul	170
RgM_RgC_type	106	RgV_sub	170
RgM_RgV_mul	171	RgV_sum	171
RgM_RgX_mul	171	RgV_sumpart	171
RgM_Rg_add	170	RgV_sumpart2	171
RgM_Rg_add_shallow	170	RgV_to_F2v	113
RgM_Rg_div	170	RgV_to_Flv	154
RgM_Rg_mul	170	RgV_to_FpV	106
RgM_Rg_sub	170	RgV_to_RgM	202
RgM_Rg_sub_shallow	170	RgV_to_RgX	201
RgM_shallowcopy	253	RgV_to_RgX_reverse	201
RgM_solve	173	RgV_to_ser	202
RgM_solve_realimag	173	RgV_to_str	239, 240
RgM_sqr	171	RgV_zc_mul	156
RgM_sub	170	RgV_zm_mul	156
RgM_sumcol	171	RgXnV_red_shallow	188
RgM_to_F2m	113	RgXn_eval	188
RgM_to_Flm	154	RgXn_exp	188
RgM_to_FpM	106	RgXn_inv	188
RgM_to_FqM	109	RgXn_inv_i	188
RgM_to_nfM	275	RgXn_mul	187, 188
RgM_to_RgXV	202	RgXn_powers	188
RgM_to_RgXX	202	RgXn_powu	188
RgM_transmul	171	RgXn_powu_i	188
RgM_transmultosym	171	RgXn_recip_shallow	187
RgM_type	106	RgXn_red_shallow	187
RgM_type2	106	RgXn_reverse	188
RgM_zc_mul	156	RgXn_sqr	187, 188
RgM_zm_mul	156	RgXn_sqrt	188
RgV_add	170	RgXQC_red	189

RgXQM_mul	189	RgX_deflate_order	184
RgXQM_red	189	RgX_degree	183
RgXQV_red	189	RgX_deriv	187
RgXQV_RgXQ_mul	189	RgX_digits	186
RgXQX_div	190	RgX_disc	187
RgXQX_divrem	190	RgX_div	185
RgXQX_mul	190	RgX_divrem	185
RgXQX_powers	190	RgX_divs	185
RgXQX_pseudodivrem	186	RgX_div_by_X_x	185
RgXQX_pseudorem	186	RgX_equal	183
RgXQX_red	189	RgX_equal_var	183
RgXQX_rem	190	RgX_even_odd	140, 183
RgXQX_RgXQ_mul	190	RgX_extgcd	186, 187
RgXQX_sqr	190	RgX_extgcd_simple	186
RgXQX_translate	190	RgX_fpnorml2	187
RgXQ_charpoly	189	RgX_gcd	186
RgXQ_inv	188	RgX_gcd_simple	186
RgXQ_matrix_pow	188	RgX_gtofp	187
RgXQ_mul	188	RgX_inflate	184
RgXQ_norm	189	RgX_integ	187
RgXQ_pow	188	RgX_isscalar	183
RgXQ_powers	188	RgX_is_FpX	114
RgXQ_powu	188	RgX_is_FpXQX	119
RgXQ_ratlift	189	RgX_is_monomial	183
RgXQ_reverse	189	RgX_is_QX	183
RgXQ_sqr	188	RgX_is_rational	183
RgXV_RgV_eval	185	RgX_is_ZX	183
RgXV_to_RgM	202	RgX_mul	185
RgXV_unscale	187	RgX_mulhigh_i	188
RgXX_to_RgM	202	RgX_muls	185
RgXY_degrees	202	RgX_mulspec	186
RgXY_swap	202	RgX_mulXn	186
RgXY_swapspec	202	RgX_mul_i	185
RgX_act_G12Q	187	RgX_mul_normalized	185
RgX_act_ZG12Q	187	RgX_neg	184
RgX_add	184	RgX_nffix	293
RgX_addmulXn	186	RgX_normalize	185
RgX_addmulXn_shallow	186	RgX_pseudodivrem	186
RgX_addspec	186	RgX_pseudorem	186
RgX_addspec_shallow	186	RgX_recip	183
RgX_blocks	183	RgX_recip_shallow	183
RgX_check_QX	181	RgX_rem	186
RgX_check_ZX	176	RgX_renormalize	183
RgX_check_ZXX	180	RgX_renormalize_lg	183
RgX_coeff	183	RgX_rescale	187
RgX_copy	183	RgX_rescale_to_int	184
RgX_cxeval	221	RgX_resultant_all	187
RgX_deflate	183	RgX_RgMV_eval	221
RgX_deflate_max	184	RgX_RgM_eval	221

RgX_RgV_eval	185	Rg_is_FF	227
RgX_RgXnV_eval	188	Rg_is_Fp	106
RgX_RgXn_eval	188	Rg_is_FpXQ	119
RgX_RgXQV_eval	189	Rg_nffix	292, 293
RgX_RgXQ_eval	188, 189	Rg_RgC_sub	170
RgX_Rg_add	184	Rg_RgX_sub	185
RgX_Rg_add_shallow	184	Rg_to_F2	154
RgX_Rg_div	185	Rg_to_F2xq	154
RgX_Rg_divexact	185	Rg_to_Fl	153, 154
RgX_Rg_eval_bk	185	Rg_to_Flxq	154
RgX_Rg_mul	185	Rg_to_Fp	106, 107
RgX_Rg_sub	185	Rg_to_FpXQ	119
RgX_Rg_type	105	Rg_to_Fq	119
RgX_rotate_shallow	184	Rg_to_RgC	202
RgX_shift	140, 184	Rg_type	104
RgX_shift_inplace	184	RM_round_maxrank	269, 289, 304
RgX_shift_inplace_init	184	rnfcomplete	273
RgX_shift_shallow	184	rnfdisc_factored	292
RgX_splitting	132, 183	rnfeltabstorel	292
RgX_sqr	185	rnfeltreltoabs	292
RgX_sqrhigh_i	188	rnfeltup	292
RgX_sqrspec	186	rnfequationall	292
RgX_sqr_i	185	rnf_build_nfabs	273
RgX_sub	184	rnf_COND	291
RgX_sylvestermatrix	208	rnf_get_absdegree	270
RgX_to_F2x	154	rnf_get_alpha	271
RgX_to_Flx	154	rnf_get_degree	270
RgX_to_FlxqX	154	rnf_get_disc	271
RgX_to_FpX	114	rnf_get_idealdisc	271
RgX_to_FpXQX	119	rnf_get_index	271
RgX_to_FqX	119	rnf_get_invzk	271
RgX_to_nfX	275	rnf_get_k	271
RgX_to_RgC	201	rnf_get_map	271, 292
RgX_to_RgV	202	rnf_get_nf	271
RgX_to_ser	202	rnf_get_nfdegree	271
RgX_to_ser_inexact	202	rnf_get_nfpol	271
RgX_translate	187	rnf_get_nfvarn	271
RgX_type	105	rnf_get_nfzk	271, 292
RgX_type2	106	rnf_get_pol	271
RgX_type3	106	rnf_get_polabs	271, 273
RgX_type_decode	105	rnf_get_varn	271
RgX_type_is_composite	105	rnf_get_zk	271
RgX_unscale	187	rnf_REL	291
RgX_val	184	rnf_zkabs	273
RgX_valrem	184	rootmod	145
RgX_valrem_inexact	184	rootmod0	145
Rg_col_ei	204	rootmod2	145
Rg_get_0	33, 105	rootsof1pow	232
Rg_get_1	105	rootsof1powinit	232

rootsof1q_cx	232	sd_echo	262
rootsof1u_cx	232	sd_factor_add_primes	262
rootsof1u_Fp	99	sd_factor_proven	262
rootsof1_cx	232	sd_format	262
rootsof1_Fl	99	sd_graphcolormap	262
rootsof1_Fp	99	sd_graphcolors	262
roots_from_deg1	209	sd_help	262
roots_to_pol	209	sd_histfile	262
roots_to_pol_r1	209	sd_histsize	262
roundr	84	sd_intarray	264
roundr_safe	85	sd_lines	262
row vector	33	sd_linewrap	262
row	253	sd_log	262
rowcopy	253	sd_logfile	262
rowpermute	254	sd_nbthreads	262
rowslice	253	sd_new_galois_format	262
rowslicepermute	254	sd_output	263
rowsplice	254	sd_parisize	263
row_i	253	sd_parisizemax	263
rtodbl	28, 199	sd_path	263
rtor	84	sd_plothsizes	263
R_abs	223	sd_prettyprinter	263
R_abs_shallow	223	sd_primelimit	263
R_from_QR	173	sd_prompt	263
		sd_prompt_cont	263
		sd_psfile	263
		sd_readline	263
		sd_realbitprecision	263
		sd_realprecision	263
		sd_recover	263
		sd_secure	263
		sd_seriesprecision	263
		sd_simplify	263
		sd_sopath	263
		sd_strictargs	263
		sd_strictmatch	263
		sd_string	264
		sd_TeXstyle	262
		sd_threadsize	263
		sd_threadsizemax	263
		sd_timer	263
		sd_toggle	263
		sd_ulong	264
		secure	53
		serchop0	203
		serchop_i	203
		sertoser	225
		ser_inv	226
		ser_isexactzero	226
S			
scalarcol	204		
scalarcol_shallow	208		
scalarmat	204		
scalarmat_s	204		
scalarmat_shallow	208		
scalarpol	203		
scalarpol_shallow	208		
scalarser	203		
scalar_Flm	109		
scalar_ZX	176		
scalar_ZX_shallow	176		
sdivsi	93		
sdivsi_rem	93		
sdivss_rem	93		
sdomain_isincl	328		
sd_breakloop	262		
sd_colors	262		
sd_compatible	262		
sd_datadir	262		
sd_debug	262		
sd_debugfiles	262		
sd_debugmem	262		

ser_normalize	226	smodis	93
ser_unscale	226	smodsi	94
setabssign	60	smodss	94
setdefault	52, 261	snm_closure	260
setexpo	30, 33, 60	sort	212
setisclone	28	sort_factor	213, 214
setlg	28, 60	split_realimag	173
setlgefint	29, 60	sprintf	39
setprecp	31, 60	sqrfrac	223
setrand	96	sqri	87
setrealprecision	231	sqrr	87
setsigne	29, 32, 33, 60	sqrs	88
settyp	28, 60	sqrti	95
setvalp	31, 33, 60	sqrtnr	231
setvarn	25, 32, 33, 60, 208	sqrtnr_abs	231
set_lex	260	sqrttr	231
set_sign_mod_divisor	285	sqrtremi	95
<i>shallow</i>	49	sqrttr_abs	231
shallowconcat	253	sqru	88
shallowconcat1	253	sstoQ	204
shallowcopy	27, 252	<i>stack</i>	13, 17
shallowextract	253	stackdummy	67, 83
shallowmatconcat	253	stack_calloc	65
shallowmatextract	253	stack_lim	23
shallowtrans	253	stack_malloc	64, 239
shiftaddress	65	stack_malloc_align	65
shiftaddress_canon	65	stack_sprintf	239
shifti	85	stack_strcat	239
shiftnl	78	stack_strdup	239
shiftnr	78	stderr	241
shiftr	85	stdout	38, 241
shiftr_inplace	85	stoi	27, 83
shift_left	85	stor	83
shift_right	85	Str	240
SIGNBITS	62	Strexpend	240
signe	29, 32, 33, 58	strftime_expand	243
SIGNnumBITS	62	<i>string context</i>	72
SIGNSHIFT	62	strntoGENstr	239
simplefactmod	145	Strprintf	39
simplify	68	Strtex	240
simplify_shallow	68	strtoclosure	260
sisfundamental	100	strtoffunction	260
sizedigit	59	strtoGENstr	239, 240
smallpolred	294	strtoi	82
smallpolred2	294	strtor	82
SMALL_ULONG	80	str_init	240
smith	302	str_printf	240
smithall	302	str_putc	240
smithclean	302	str_puts	240

subgroups_tableset	236	trans_eval	232
subiu	90	traverseheap	69
subll	77	tridiv_bound	159
subllx	77	trivialgroup	235
submulii	91	trivial_fact	204
submuliu	91	truecoef	224, 226
submuliu_inplace	91	truecoeff	61
subresex	216	truedivii	91
subui	90	truedivis	91
subuu	90	truedivsi	91
sumdedekind	100	truedvmdii	93
sumdedekind_coprime	100	truedvmdis	93
sumdigitsu	100	truedvmdsi	93
switchin	54	trunc2nr	85
switchout	241	trunc2nr_lg	85
szeta	231	truncr	84
		trunc_safe	84
		TWOPOTBITS_IN_LONG	62
		typ	28, 57
		TYPBITS	62
		type number	28
		type	14
		type_name	57
		TYPnumBITS	62
		TYPSHIFT	62
		typ_BNF	267
		typ_BNR	267
		typ_NF	267
		t_CLOSURE	34
		t_COL	33
		t_COMPLEX	31
		t_ELL_Fp	311
		t_ELL_Fq	311, 317
		t_ELL_NF	311
		t_ELL_Q	311
		t_ELL_Qp	311
		t_ELL_Rg	312
		t_ERROR	34
		t_FFELT	31
		t_FF_F2xq	31
		t_FF_Flxq	31
		t_FF_FpXQ	31
		t_FRAC	31
		t_INFINITY	34
		t_INT	29
		t_INTMOD	31
		t_LIST	34
		t_MAT	34
		t_PADIC	31
T			
tablemul	276		
tablemulvec	276		
tablemul_ei	276		
tablemul_ei_ej	276		
tablesearch	214		
tableset_find_index	236		
tablesqr	276		
teichmullerinit	149		
term_color	242		
term_get_color	242		
texe	242		
theta_get_an	327		
theta_get_bitprec	327		
theta_get_K	327		
theta_get_m	328		
theta_get_R	327		
theta_get_sqrtN	328		
theta_get_tdom	328		
threads	339		
timer	41		
timer2	41		
timer_delay	41		
timer_get	41		
timer_printf	41		
timer_start	41		
togglesign	60		
togglesign_safe	60, 165, 166		
to_famat	174, 277		
to_famat_shallow	174, 278		
Tp_to_FF	227		

t_POL	32	uprecprime	162
t_POLMOD	32	uprime	162
t_QFI	33	uprimepi	162
t_QFR	33	upr_norm	281
t_QUAD	31	usqrt	157
t_REAL	30	usqrtn	157
t_RFRAC	33	usumdivk_fact	100
t_SER	33	usumdiv_fact	100
t_STR	34	utoi	83
t_VEC	33	utoineg	83
t_VECSMALL	34	utoipos	83

U

uabsdiviu_rem	93	utor	83
uabsdivui_rem	93	uu32toi	26, 84
udivuu_rem	93	uu32toineg	84
ugcd	95	uutoi	84
ugcdiu	95	uutoineg	84
ugcdui	96	u_chinese_coprime	145
uislucasp	100	u_forprime_arith_init	164
uisprime	163	u_forprime_init	43, 164
uisprimepower	100	u_forprime_next	43, 164
uisprime_101	163	u_forprime_restrict	164
uisprime_661	163	u_lval	86
uissquare	157	u_lvalrem	86
uissquareall	157	u_lvalrem_stop	86
uissquarefree	100	u_ppo	160
uissquarefree_fact	100	u_pval	86
uis_357_power	100	u_pvalrem	86
uis_357_powermod	100	u_sumdedekind_coprime	100
ulcm	96		
ulogint	95		
ulogintall	95		
ulong	49		
ULONG_MAX	62		
umodi2n	94		
umodiu	94		
umodsu	94		
umodui	94		
umuluu_le	91		
umuluu_or_0	91		
unegisfundamental	100		
unextprime	162		
unsetisclone	28		
uordinal	239		
uposisfundamental	100		
upowuu	95		
upper_to_cx	199		

V

vali	85
valp	31, 33, 58
VALPBITS	62
VALPnumBITS	62
vals	85
varargs	25
varhigher	35
variable (priority)	34
variable (temporary)	36
variable (user)	35
variable number	32, 35, 72
varlower	35
varn	32, 33, 35, 58
VARNBITS	62
varncmp	35
varnmax	37
varnmin	37
VARNnumBITS	62

ZabM_inv_ratlift	277	zero_zv	169
ZabM_ker	276	zero_zx	182
ZabM_pseudoinv	276	ZGCs_add	175
zCs_to_ZC	175	ZGC_G_mul	174
ZC_add	164	ZGC_G_mul_inplace	174
ZC_copy	165	ZGC_Z_mul	174
ZC_hnfrem	305	ZG_add	174
ZC_hnfremdiv	304	ZG_G_mul	174
ZC_is_ei	168	ZG_mul	174
ZC_lincomb	165	ZG_neg	174
ZC_lincomb1_inplace	165	ZG_normalize	174
ZC_lincomb1_inplace_i	165	ZG_sub	174
ZC_neg	165	ZG_Z_mul	174
ZC_nfval	284	Zideallog	307
ZC_nfvalrem	284	zidealstar	294
ZC_prdvd	284	zidealstarinit	294
ZC_Q_mul	167	zidealstarinitgen	294
ZC_reducemodlll	305	zkchinese	280
ZC_reducemodmatrix	305	zkchinese1	280
ZC_sub	164	zkchineseinit	280
zc_to_ZC	155	zkC_multable_mul	275
ZC_union_shallow	166	zkmodprinit	283
ZC_ZV_mul	165	zkmultable_capZ	276
ZC_Z_add	165	zkmultable_inv	276
ZC_Z_div	165	zk_inv	276
ZC_Z_divexact	165	zk_multable	275, 279
ZC_z_mul	156	zk_scalar_or_multable	276, 281
ZC_Z_mul	165	zk_to_Fq	283
ZC_Z_sub	165	zk_to_Fq_init	283
zerocol	203	zlm_echelon	152
zeromat	204	Zlm_gauss	152
zeromatcopy	204	ZMrow_ZC_mul	167
zeropadic	203	zMs_to_ZM	175
zeropadic_shallow	208	zMs_ZC_mul	175
zeropol	203	ZMV_to_FlmV	169
zeroser	203	ZMV_to_zmV	169
zerovec	203	zmV_to_ZMV	169
zerovec_block	204	ZM_add	166
zero_F2m	113	ZM_charpoly	167
zero_F2m_copy	113	ZM_copy	166
zero_F2v	113	zm_copy	169
zero_F2x	139	ZM_det	167
zero_Flm	111	ZM_detmult	167
zero_Flm_copy	111	ZM_det_triangular	168
zero_Flv	111	ZM_diag_mul	166
zero_Flx	131	ZM_equal	166
zero_FlxC	133	ZM_equal0	166
zero_FlxM	133	ZM_famat_limit	278
zero_zm	169	ZM_gauss	168

ZM_hnf	299, 302	ZM_snfclean	301
ZM_hnfall	299, 300, 302	ZM_snf_group	301, 302
ZM_hnfall_i	300	ZM_sqr	166
ZM_hnfcenter	300	ZM_sub	166
ZM_hnfdivrem	305	ZM_supnorm	167, 220
ZM_hnflll	300	ZM_togglesign	166
ZM_hnfmod	299, 302	ZM_to_F2m	113
ZM_hnfmodall	299, 300	ZM_to_Flm	154
ZM_hnfmodall_i	300	zm_to_Flm	155
ZM_hnfmodid	299, 302	ZM_to_zm	154
ZM_hnfmodprime	299	zm_to_ZM	155
ZM_hnfperm	300	zm_to_zxV	156
ZM_hnfrem	305	ZM_transmul	167
ZM_hnf_knapsack	300	ZM_transmultosym	166
ZM_imagecompl	167	zm_transpose	169
ZM_incremental_CRT	146	ZM_zc_mul	156
ZM_indeximage	167	ZM_ZC_mul	166
ZM_indexrank	167	zm_zc_mul	169
ZM_init_CRT	146	ZM_zm_mul	156
ZM_inv	167	ZM_ZX_mul	166
ZM_inv_ratlift	167	ZM_Z_div	167
ZM_isdiagonal	168	ZM_Z_divexact	167
ZM_ishnf	168	ZM_Z_mul	166
ZM_isidentity	168	zncharcheck	306
ZM_isscalar	168	zncharconj	306
ZM_ker	167	znchardiv	306
ZM_lll	303	znchareval	307
ZM_lll_norms	304	zncharker	307
ZM_max_lg	167	zncharm	307
ZM_mul	166	zncharorder	307
zm_mul	169	zncharpow	307
ZM_multosym	166	znchar_quad	307
ZM_mul_diag	166	znconreyfromchar	307
ZM_neg	166	znconreyfromchar_normalized	307
ZM_nm_mul	156	znconreylog_normalize	307
ZM_nv_mod_tree	147	znconrey_check	307
ZM_permanent	167	znconrey_normalized	307
zm_permanent	169	znstar_get_conreycyc	272
ZM_pow	167	znstar_get_conreygen	272
ZM_powu	167	znstar_get_cyc	272
ZM_pseudoinv	167	znstar_get_faN	272
ZM_Q_mul	167	znstar_get_gen	272
ZM_rank	167	znstar_get_N	272
ZM_reducemodlll	305	znstar_get_no	272
ZM_reducemodmatrix	305	znstar_get_pe	272
zm_row	169	znstar_get_U	272
ZM_snf	301	znstar_get_Ui	272
ZM_snfall	301	Zn_ispower	99
ZM_snfall_i	301	Zn_issquare	99

Zn_sqrt	98	ZV_content	165
ZpMs_ZpCs_solve	176	zv_content	169
ZpM_echelon	152	zv_copy	169
ZpXQM_prodFrobenius	151	ZV_dotproduct	165
ZpXQX_digits	152	zv_dotproduct	169
ZpXQX_divrem	152	ZV_dotsquare	165
ZpXQX_liftfact	151	ZV_dvd	165
ZpXQX_liftroot	151, 152	ZV_equal	164
ZpXQX_liftroot_vald	152	zv_equal	169
ZpXQX_roots	152	ZV_equal0	164
ZpXQ_div	151	zv_equal0	169
ZpXQ_inv	151	ZV_extgcd	96, 165
ZpXQ_invlift	151	ZV_indexsort	166
ZpXQ_log	151	ZV_isscalar	172
ZpXQ_sqrt	151	ZV_lval	87
ZpXQ_sqrtnlift	151	ZV_lvalrem	87
ZpX_disc_val	150	ZV_max_lg	165
ZpX_Frobenius	150	zv_neg	168
ZpX_gcd	150	ZV_neg_inplace	165
ZpX_liftfact	150	zv_neg_inplace	168
ZpX_liftroot	149, 150, 152	ZV_nv_mod_tree	147
ZpX_liftroots	150	ZV_prod	165
ZpX_monic_factor	151	zv_prod	169
ZpX_reduced_resultant	150	ZV_producttree	147, 148
ZpX_reduced_resultant_fast	150	zv_prod_Z	169
ZpX_resultant_val	150	ZV_pval	87
ZpX_roots	150	ZV_pvalrem	87
ZpX_ZpXQ_liftroot	151	ZV_search	166
ZpX_ZpXQ_liftroot_ea	151	zv_search	169
Zp_issquare	99	ZV_snfall	301
Zp_sqrt	149	ZV_snf_group	301
Zp_sqrtnlift	149	ZV_snf_trunc	301
Zp_sqrtnlift	149	ZV_sort	165, 166
Zp_teichmuller	149	ZV_sort_inplace	165
ZqX_liftfact	152	ZV_sort_uniq	165
ZqX_liftroot	152	ZV_sum	165
ZqX_roots	152	zv_sum	169
Zq_sqrtnlift	151	ZV_togglesign	165
zvV_equal	169	ZV_to_F2v	113
ZV_abscmp	164	ZV_to_Flv	154
ZV_allpnqn	96	zv_to_Flv	155
ZV_cba	160	ZV_to_nv	154
ZV_cba_extend	160	ZV_to_zv	154
ZV_chinese	147	zv_to_ZV	155
ZV_chinesetree	148	zv_to_zx	156
ZV_chinese_center	147	ZV_union_shallow	166
ZV_chinese_tree	148	ZV_zc_mul	156
ZV_cmp	164, 214	ZV_zMs_mul	175
zv_cmp0	169	zv_ZM_mul	156

ZV_ZM_mul	167	ZX_equal	176, 179
ZV_Z_dvd	87	ZX_equal1	176
zv_z_mul	169	ZX_eval1	178
ZXC_nv_mod_tree	147	ZX_factor	178
ZXM_incremental_CRT	146	ZX_gcd	177
ZXM_init_CRT	146	ZX_gcd_all	177
ZXM_nv_mod_tree	147	ZX_graeffe	178
ZXn_mul	180	ZX_incremental_CRT	146
ZXn_sqr	180	ZX_init_CRT	146
ZXQM_mul	180	ZX_is_irred	178
ZXQM_sqr	180	ZX_is_monic	176
ZXQX_dvd	186	ZX_is_squarefree	178
ZXQX_mul	180	ZX_lval	87
ZXQX_sqr	180	ZX_lvalrem	87
ZXQ_charpoly	179	ZX_max_lg	176
ZXQ_mul	179	ZX_mod_Xnm1	177
ZXQ_sqr	179	ZX_mul	177, 181
ZXT_remi2n	179	ZX_mulspec	177
ZXT_to_FlxT	154	ZX_mulu	177
ZXV_dotproduct	179	ZX_neg	176
ZXV_equal	179	ZX_nv_mod_tree	147
ZXV_remi2n	179	ZX_primitive_to_monic	177
ZXV_to_FlxV	154	ZX_pval	87
ZXV_Z_mul	179	ZX_pvalrem	87
ZXXT_to_FlxXT	154	ZX_Q_mul	178
ZXXV_to_FlxXV	154	ZX_Q_normalize	178, 285
ZXX_max_lg	180	ZX_radical	177
ZXX_mul_Kronecker	181	ZX_rem	177
ZXX_nv_mod_tree	147	ZX_remi2n	177
ZXX_renormalize	180	ZX_renormalize	176
ZXX_sqr_Kronecker	181	zx_renormalize	182
ZXX_to_F2xX	142	ZX_rescale	178
ZXX_to_FlxX	154	ZX_rescale2n	178
zxX_to_FlxX	154	ZX_rescale_lt	178
zxX_to_Kronecker	136	ZX_resultant	178
ZXX_to_Kronecker	180	zx_shift	182
ZXX_to_Kronecker_spec	180	ZX_shifti	177
ZXX_Z_add_shallow	180	ZX_sqr	177, 181
ZXX_Z_divexact	180	ZX_sqrspec	177
ZXX_Z_mul	180	ZX_squff	178
ZX_add	176	ZX_sturm	179
ZX_compositum_disjoint	179	ZX_sturmpart	179
ZX_content	177	ZX_sub	176
ZX_copy	176	ZX_to_F2x	139
ZX_deflate_max	178	ZX_to_Flx	154
ZX_deflate_order	178	zx_to_Flx	155
ZX_deriv	178	ZX_to_monic	177
ZX_disc	178	zx_to_zv	156
ZX_div_by_X_1	177	zx_to_ZX	155

ZX_translate	178	Z_ppgle	160
ZX_unscale	178	Z_ppio	160
ZX_unscale2n	178	Z_ppo	160
ZX_unscale_div	178	Z_pval	86
ZX_Uspensky	178	z_pval	86
ZX_val	177	Z_pvalrem	86
ZX_valrem	177	z_pvalrem	86
ZX_Zp_root	150	Z_smoothen	158
ZX_ZXY_resultant	179	Z_to_F2x	139
ZX_ZXY_rnfequation	179	Z_to_Flx	156
ZX_Z_add	176	Z_to_FpX	116
ZX_Z_add_shallow	176	Z_to_perm	234
ZX_Z_divexact	177	Z_ZC_sub	165
ZX_Z_mul	177	Z_ZV_mod	146
ZX_Z_normalize	177, 178	Z_ZV_mod_tree	147
ZX_Z_sub	176	Z_ZX_sub	177
ZX_z_unscale	178		
Z_cba	160		
Z_chinese	145	_evalexpo	59
Z_chinese_all	145	_evallg	59
Z_chinese_coprime	145	_evalprecp	59
Z_chinese_post	145	_evalvalp	59
Z_chinese_pre	145		
Z_content	216		
Z_ECM	159		
Z_factor	158, 159, 160		
Z_factor_limit	158, 160		
Z_factor_listP	158		
Z_factor_until	158		
Z_FF_div	228		
Z_incremental_CRT	146		
Z_init_CRT	146		
Z_isanypower	157, 160		
Z_isfundamental	162		
Z_isplay2	157		
Z_isplay	157		
Z_isplayall	157		
Z_issmooth	158		
Z_issmooth_fact	158		
Z_issquare	157		
Z_issquareall	157		
Z_issquarefree	162		
Z_lval	86		
z_lval	86		
Z_lvalrem	86		
z_lvalrem	86		
Z_lvalrem_stop	86		
Z_nv_mod	147		
Z_pollardbrent	159		